UNIT 4 SIMILAR FIGURES

Unit outcomes

After Completing this unit, you should be able to:

- know the concept of similar figures and related terminologies.
- understand the condition for triangles to be similar.
- apply tests to check whether two given triangles are similar or not.

Introduction

You may see the map of Ethiopia either in smaller or larger size, but have you asked yourself about the difference and likness of these maps? In geometery this concept is described by "similarity of plane figures" and you learn this concept here in this unit. You begin this by studying similarity of triangles and how to compare their areas and perimeters.

4.1 Similar Plane Figures

Activity 4.1

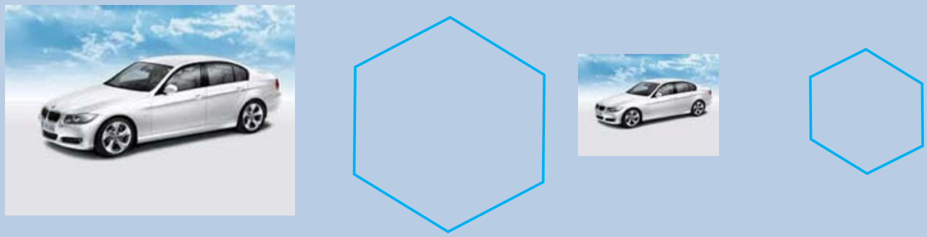
Discuss with your teacher

1. Which of the following maps are similar?

(a) (b) (c) (d)

Figure 4.1

2. Which of the following pictures or polygons are similar?



(a) (b) (c) (d)

Figure 4.2

3. Which of the following polygons are similar?

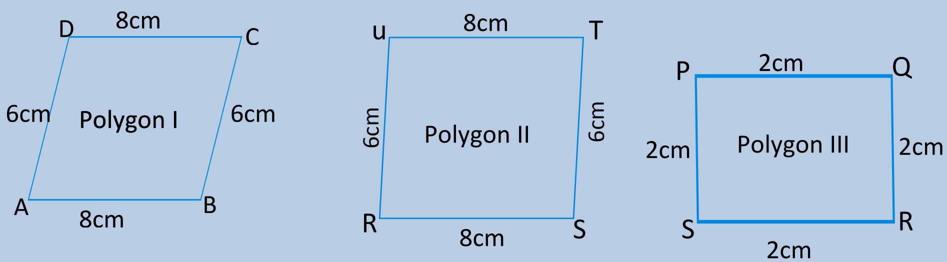


Figure 4.3

Similar geometric Figures are figures which have exactly the same shape. See Figure 4.4, each pair of figures are similar.

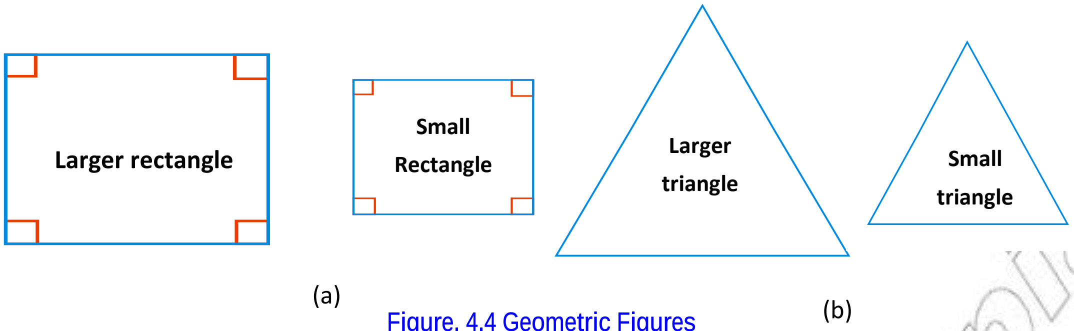


Figure. 4.4 Geometric Figures

Therefore, geometric figures having the same shape, equal corresponding angles and corresponding sides are proportional are called **similar figures**.

4.1.1 Illustration and Definition of Similar Figures

Group Work 4.1

1. A square and a rectangle have corresponding angles congruent.

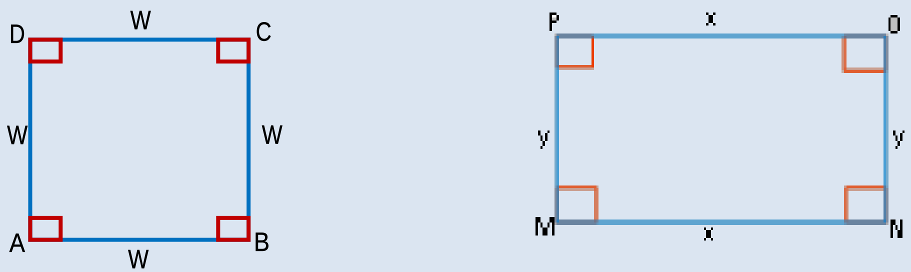
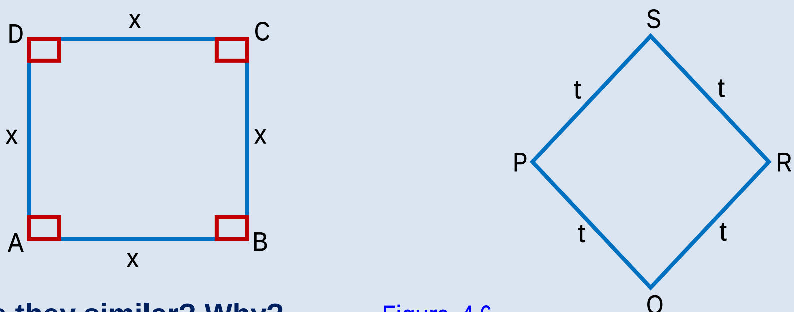


Figure. 4.5

Are they similar? Why?

2. A square and a rhombus have the corresponding sides proportional.



Are they similar? Why?

Figure. 4.6

3. Which members of these families of shape are similar:
- **a) squares**
- **b) rectangles**
- **c) Parallelograms**
- **d) circles**
- **e) equilateral triangles**
- **f) isosceles triangles**
- **g) regular hexagons**
- **h) trapeziums**

4. Which members of these families of solid shape are similar:
- **a) cubes**
- **b) cuboids**
- **c) spheres**
- **d) tetrahedrons**
- **e) pyramids**
- **f) cones**

Definition 4.1: **Two polygons are similar, if:**

i. their corresponding sides are proportional.

ii. their corresponding angles are congruent (equal).

Example1: Which of the following polygons are similar? Which are not? state the reason.

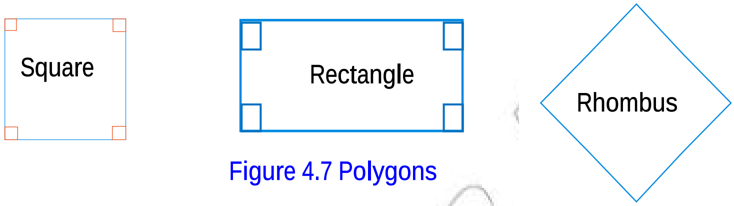


Figure 4.7 Polygons

Solution: From the given Figure 4.7 above

a. The square and the rectangle are not similar because their corresponding angles are congruent but their corresponding sides are not proportional.
b. The square and the rhombus are not similar because their corresponding sides are proportional but their corresponding angles are not congruent.
c. The rectangle and the rhombus are not similar because their corresponding angles are not congruent and corresponding sides are not proportional.

Example2: Tell whether each pairs of in Figures 4.8 is similar or not.

(a) (b)

Figure 4.8 Polygons

Solution:

a. Paris of the quadrilaterals have the same shape and angles but have not the same size. Therefore, they are not similar.

b. Paris of the quadrilaterals have not the same shape. Therefore, they are not similar.

Exercise 4A

Which of the following figures are always similar?

a. Any two circles.
b. Any two line segments.
c. Any two quadrilaterals.
d. Any two isosceles triangles.
e. Any two squares.
f. Any two rectangles.
g. Any two equilateral triangles.

4.1.2 Scale Factors and Proportionality

Activity 4.2

Discuss with your friends.

1. Have you observed what they do in the film studio?
2. How do you see films in the cinema house?
3. How are the pictures enlarged on the cinema screen?
4. What is meant by scale factor?
5. What is meant by proportional sides of similar figures?

To discuss Activity 4.2, it is important to study central enlargement (central stretching). **Central enlargement** which is either increases or decreases the size of figures with out affecting their shapes.

Under an enlargement

1. Lines and their images are parallel.
2. Angles remain the same.
3. All lengths are increased or decreased in the same ratio.

Positive enlargement

In Figure 4.9 triangle $A_1B_1C_1$ is the image of triangle ABC under enlargement. O is the centre of enlargement and the lines AA_1, CC_1 and BB_1 when produced must all pass through O.

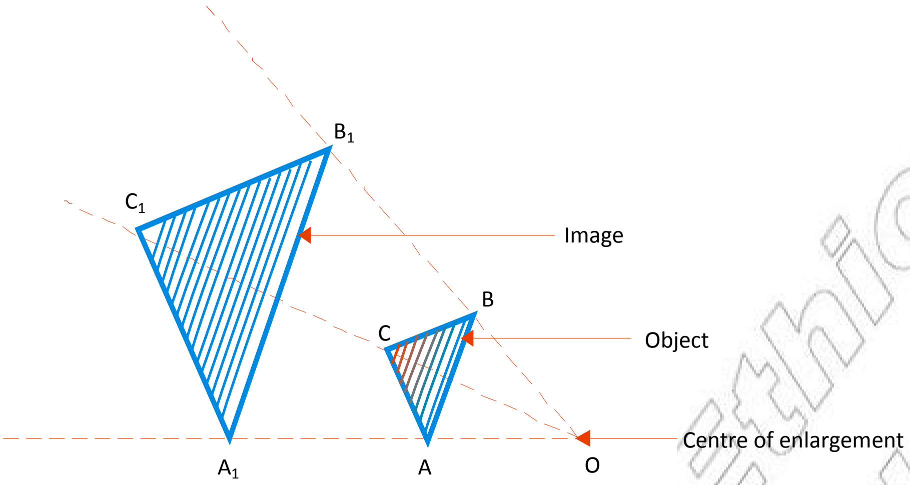


Figure 4.9 Positive enlargement

The enlargement in Figure 4.9 is called a **positive enlargement**, because both the object and its image are on the same side of the centre of enlargement. Also the image is further from O than the object.

Note: - AB is parallel to A_1B_1.
- BC is parallel to B_1C_1.
- AC is parallel to A_1C_1.
- Angle BAC≅ angle $B_1 A_1 C_1$.
- Angle ABC ≅ angle $A_1 B_1 C_1$.
- Angle BCA≅angle $B_1C_1 A_1$.

? "In Figure 4.9 above the object and the image are similar why?"

Example3: Given the rectangle, ABCD and A as the centre of enlargement. Draw the image AB'C'D' after enlargement of each side of ABCD twice.

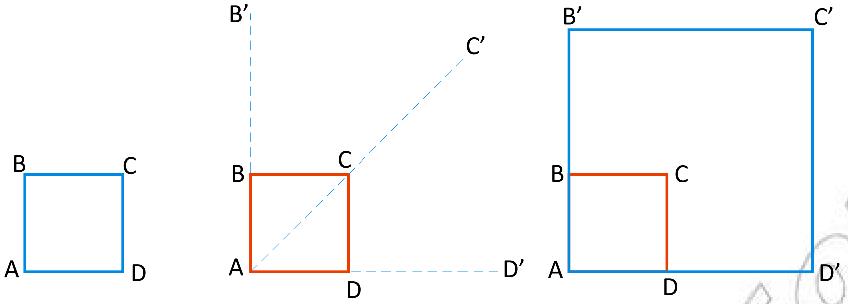


Figure. 4.10

Solution:

Point A is the centre of enlargement and is fixed. **A' is at A**. Since each sides of A'B'C'D' enlarged twice of each side of ABCD,

$$\frac{AB}{A'B'}=\frac{AD}{A'D'}=\frac{AC}{A'C'}=\frac{DC}{D'C'}=\frac{1}{2}$$ or A'B'=2AB, A'D'=2AD A'C'=2AC and, D'C' = 2DC. The number 2 in this equation is called **the constant of proportionality or scale factor**.

? "Can you define a scale factor based on example 3 above?"

Definition 4.2: Scale factor – the ratio of corresponding sides usually expressed numerically so that:

$$\text{Scale factor}=\frac{\text{length of line segment on the enlargement}}{\text{length of line segment on the original}}$$

Notation of scale factor = K

Example 4: The vertices of triangle ABC have co-ordinates A(2,1), B(4,1) and C(3,4). Find the co-ordinates of triangle $A_1B_1C_1$ after an enlargement, scale factor 2, with centre at O.

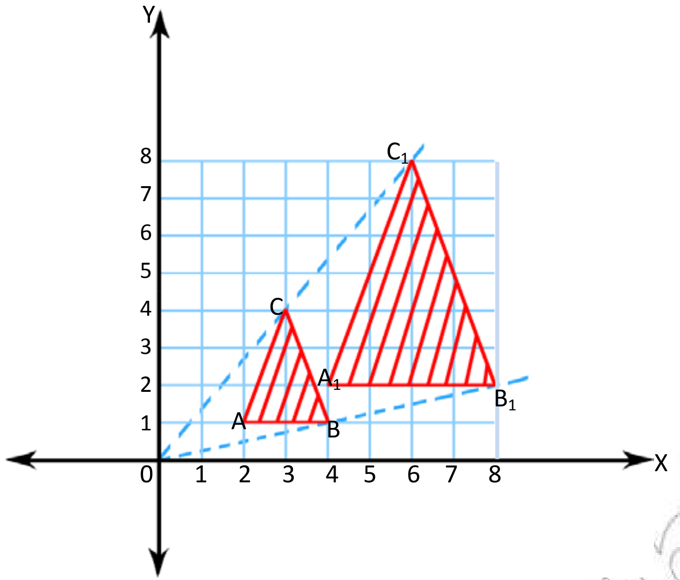


Figure. 4.11

Solution:

In Figure 4.11 you can see the object, triangle ABC and its image under enlargement, triangle $A_1B_1C_1$ with co-ordinates: A_1 (4,2), B_1 (8,2) C_1 (6,8).

Example 5: Give the shape PQRS and point O. Draw the image P'Q'R'S' after enlargement of each side of PQRS twice.

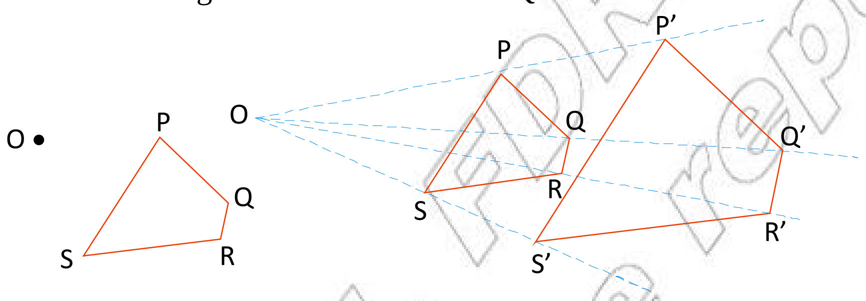


Figure 4.12

Solution: Join O to P, Q, R and S. Since each sides of P' Q' R' S' enlarged twice of each side of PQRS.

$$\frac{OP}{OP'} = \frac{OQ}{OQ'} = \frac{OR}{OR'} = \frac{OS}{OS'} = \frac{1}{2} \quad \text{or}$$

OP'=2OP, OQ'= 2OQ, OR'=2OR and OS'=2OS

Hence, the number 2 is called **the constant of proportionality** or **scale factor.**

? Can you define a central enlargement (central stretching) in your own word?

Definition 4.3: A mapping which transform a figure following the steps given below is called *central enlargement.*

Step i: Mark any point O. This point O is called center of the central enlargement.

Step ii: Fix a number K. This number K is the constant of proportionality.

Step iii: Determine the image of each point A such as A' such that A'O=KAO.

Step iv: The image of point O is itself.

According to the definition 4.3 when you find the image of a plane figure,

i. If $K > 1$, the image figure is larger than the object figure.

ii. If $0 < K < 1$, the image figure is smaller than the object figure.

iii. If K=1, the image figure is congruent to the object figure.

Example 6: Enlarge triangle PQR by scale factor 3 and O is the centre of enlargement as shown below.

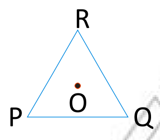


Figure 4.13

Solution:

Copy triangle PQR and the point O inside the triangle.

Step i: Make point P', R' and Q' on $\overrightarrow{OP}, \overrightarrow{OQ}$ and $\overrightarrow{OR}$ such that

P'O = 3PO, Q′O = 3QO and

R′O = 3RO see Figure 4.14 to the right.

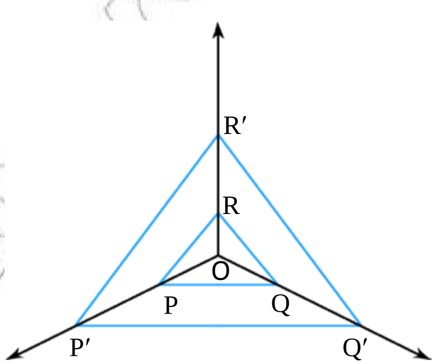


Figure 4.14

Stepii: Join the points P',Q', R' with line segment to obtain △P'Q'R' (which is the required Figure).

Exercise 4B

1. Draw the image of the shape KLMN after an enlargement by scale factor $\frac{1}{2}$ with center O. Label the image K' L' M' N'.

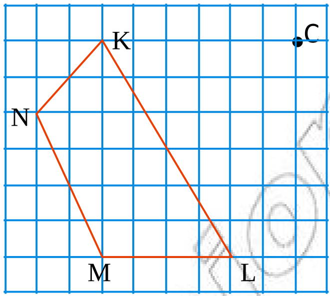


Figure 4.15

2. Work out the scale factor of the enlargement that takes in Figure 4.16, triangle ABC on the triangle LMN.

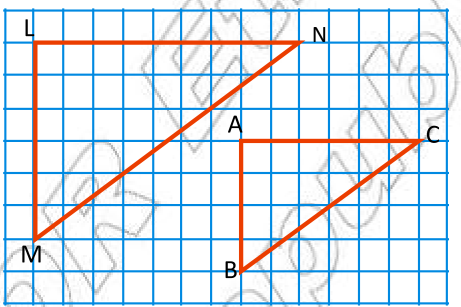


Figure 4.16

3. Copy the Figure 4.17 below. With O as centre, draw the image of the shaded shape after enlargement by:

   a. scale factor $\frac{1}{4}$

   b. scale factor $\frac{3}{4}$

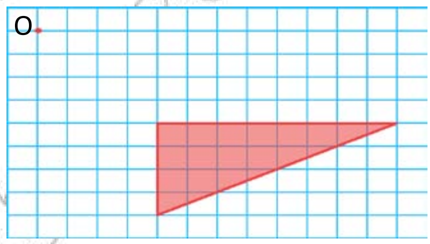


Figure 4.17

4.2 Similar Triangles

4.2.1 Introduction to Similar Triangles

Group Work 4.2

1. Consider Figure 4.18 below:

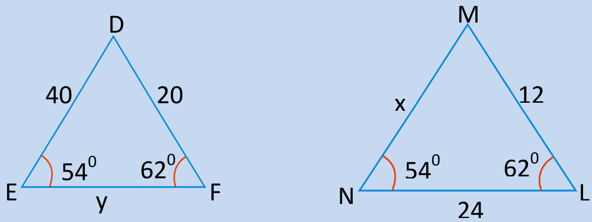


Figure 4.18

a. Are ΔDEF similar to ΔMNL? Why?

b. If ΔDEF similar to ΔMNL then find the value of X and Y.

2. Consider Figure 4.19 below:

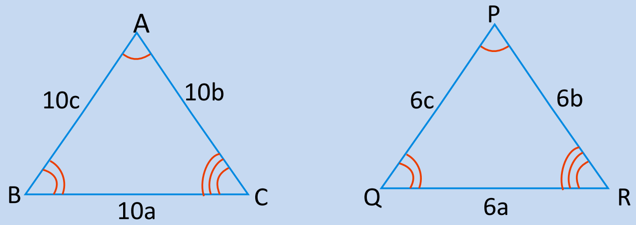


Figure 4.19

a. ΔABC is similar to ΔPQR. Explain the reason.

b. ΔABC is not similar to ΔPRQ. Explain the reason.

3. ΔXYZ is given such that ΔDEF similar ΔXYZ. Find XY and YZ when the scale factor from ΔXYZ to ΔDEF is 6 and DE=7, EF=12 and XZ=36.

You have define similar polygon in section 4.1.1. You also know that any polygon could be dived into triangles by drawing the diagonals of the polygon. Thus the definition you gave for similar polygons could be used to define similar triangles.

Definition 4.4: ΔABC is similar to ΔDEF, if

i. their corresponding sides are proportional.

ii. their corresponding angles are congruent.

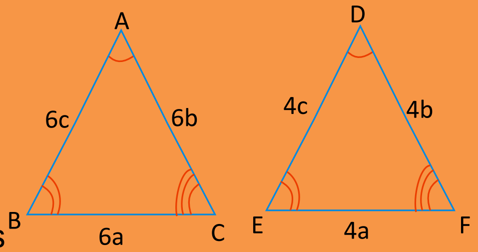


Figure. 4.20

That is symbolically:

ΔABC~ΔDEF if and only if

1. ∠A≡∠D
2. ∠B≡∠E
3. ∠C≡∠F

} corresponding angles are congruent

4. $\frac{AB}{DE}=\frac{BC}{EF}$
5. $\frac{BC}{EF}=\frac{AC}{DF}$
6. $\frac{AB}{DE}=\frac{AC}{DF}$

} corresponding sides are proportional

or the above three facts 4, 5 and 6 can be summarized as

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}=K$$

Note: From definition (4.4) of similar triangles it is obvious that similarity is a transitive relation. That is: If ΔABC~ ΔDEF and ΔDEF ~ΔXYZ, then ΔABC~ΔXYZ.

Example 7: Let ΔABC~ΔDEF. As shown in Figure 4.21 below. Find

a. m(∠F) b. m(∠E) c. the length of BC.

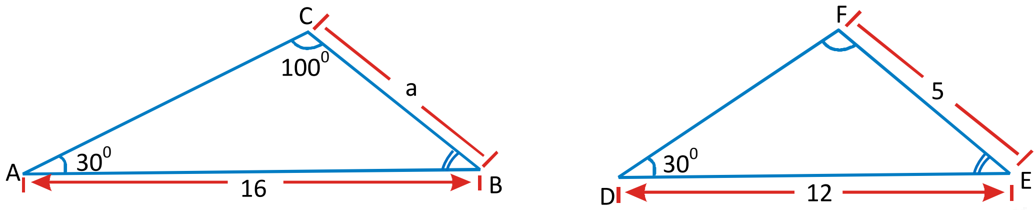


Figure 4.21

Solution: Given ΔABC ~ ΔDEF …. By definition 4.4

a. ∠A≡∠D if and only if m (∠A) = m (∠D) = 30°

∠B ≡ ∠E .... Marked angels

Therefore, ∠C ≡ ∠ F if and only if m(∠C) = m (∠F) = 100°

Therefore, m (∠F) = 100°

b. m(∠A) + m(∠B) + m (∠C) = 180° ...Angle sum theorem

30° + m (∠B) +100^0 = 180° .... Substitution

m (∠B) = 180°- 130° = 50°

Therefore, m (∠E) = 50°.

c. Given Δ ABC~ΔDEF

implies $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ .... Definition 4.4

$\frac{16}{12} = \frac{a}{5}$ .... Using the 1st and 2nd proportions

12a = 80 .... Cross multiplication

$a = \frac{80}{12}$ ....Dividing both sides by 12

Therefore, the length of BC=$\frac{80}{12}$ unit.

Example 8: In Figure 4.22 below, show that ΔABC and ΔLMN are similar.

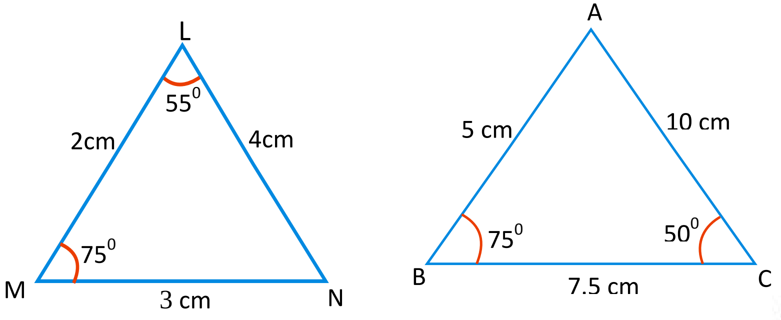


Figure 4.22

Solution: You begin by finding the unknown angels in the triangles. You know the size of two angles in each triangle and you also know that the sum of the angles of a triangle is 180^0. Therefore, it is easy to calculate the size of the unknown angles.

In ΔABC, m(∠ABC)+m(∠BCA)+m(∠CAB)=180°.... Why?

$\Rightarrow 75^0+50^0+m(\angle CAB)=180^0$ .... Substitution

$\Rightarrow m(\angle CAB)=180^0-125^0$

$\Rightarrow m(\angle CAB)=55^0$

In ΔLMN, m(∠LMN)+m(∠MNL)+m(∠NLM)=180°....Why?

$\Rightarrow 75^0+m(\angle MNL)+55^0=180^0$ .... Substitution

$\Rightarrow m(\angle MNL)=180^0-130^0$

$\Rightarrow m(\angle MNL)=50^0$

The corresponding angles are equal, to show the corresponding sides are in the same ratio. Let us check whether the corresponding sides are proportional or not. In short let us check.

ΔABC~ΔLMN

$$\Rightarrow \frac{AB}{LM}=\frac{BC}{MN}=\frac{AC}{LN}=K \text{ (constant of proportionality)}$$

$$\frac{5cm}{2cm}=\frac{7.5cm}{3cm}=\frac{10cm}{4cm}=2.5$$

The corresponding sides are proportional with constant of proportionality equals 2.5. Therefore ΔABC~ΔLMN….By definition 4.4.

Example 9: In Figure 4.23, ΔABC~ΔXYC. If CX=6, CY=5, AX= 3 and AB= 8, find XY.

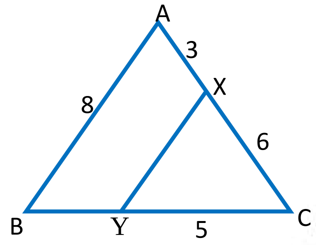


Figure 4.23

Solution:

By definition of similar triangles, we have:

$$\frac{AB}{XY}=\frac{BC}{YC}=\frac{AC}{XC}=K \text{ (constant proportionality)}$$

Thus $\frac{AB}{XY}=\frac{AC}{XC}$ .... Using the 1st and 3rd proportions

$\Rightarrow \frac{8}{XY}=\frac{AX+XC}{XC}$ since AX+XC=3+6=9

$\Rightarrow \frac{8}{XY}=\frac{9}{6}$

$\Rightarrow 9XY=48$ .... Cross multiplication

$\Rightarrow XY = \frac{48}{9}$ ....Dividing both sides by 9.

Example 10: If ΔABC≡ΔDEF, then

a. Is ΔABC~ΔDEF?

b. Justify your answer.

Solution:

a. Yes

b. Suppose ΔABC ≡ ΔDEF are as shown in Figure 4.24 below:

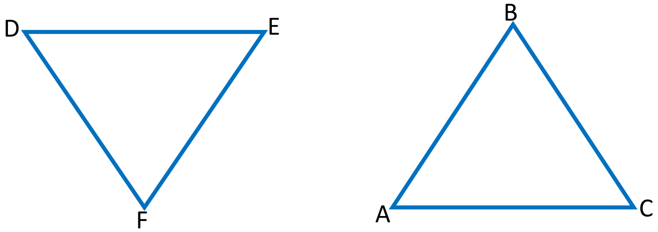


Figure 4.24

Then i. ∠ABC≡∠DEF

∠BCA≡∠EFD and

∠CAB≡∠FDE

Hence corresponding angles are congruent.

ii. $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$ and $\overline{AC} \equiv \overline{DF}$ implies

AB=DE, BC=EF, and AC=DF. Thus

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 1$$

Hence the corresponding sides are proportional. Since (from (i) and (ii))the corresponding sides are proportional with constant of proportionality 1, and also the corresponding angles are congruent, then the triangles are similar by definition 4.4. From example 10 above you can make the following generalization.

Note: Congruence is a similarity where the constant of proportionality is 1. This general fact is equivalent of the statement given below. If two triangles are congruent, then they are similar.

Exercise 4C

1. If ΔABC~ ΔB'A'C', what are the pairs of corresponding angles and the pairs of corresponding sides?
2. If ΔABC~ΔA'B'C' and AC=20cm, A'C'=15cm, B'C'=12cm and A'B'=9cm, find the lengths of the other sides of ΔABC.

3. The sides of a triangle are 4cm, 6cm, and a cm respectively. The corresponding sides of a triangle similar to the first triangle are b cm, 12 cm and 8 cm respectively. What are the lengths a and b?
4. Are two similar triangles necessarily congruent? Why?
5. What is the length of the image of a 20cm long segment after central stretching with a scale factor $\frac{1}{2}$?
6. If $\Delta DEF \sim \Delta KLM$ such that DE =(2x + 2)cm, DF = (5x – 7)cm, KL = 2cm, KM = 3cm and EF= 10cm, then find LM.
7. In Figure 4.25 if $\Delta XYZ \sim \Delta WYP$, express d in terms of a, b and c.

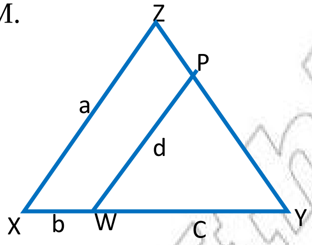


Figure 4.25

8. In Figure 4.26 below, if$\Delta ABC \sim \Delta XBZ$ with XB=6cm, BZ=5cm, CX=8cm and AC=7cm.
What is the length of
a. $\overline{BC}$?   b. $\overline{XZ}$?

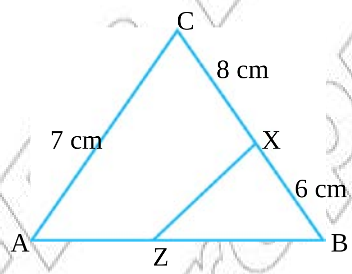


Figure. 4.26

Challenge Problems

9. Write down a pair of similar triangles in Figure 4.27 to the right. Find CD and AC, if AE is parallel to BD.

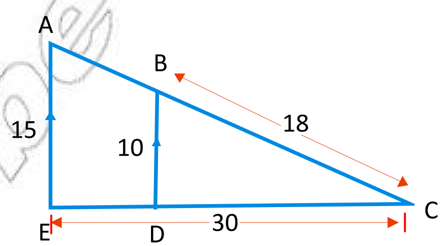


Figure. 4.27

4.2.2 Tests for Similarity of Triangles (SSS, SAS and AA)

Activity 4.3

Discuss with your teacher before starting the lesson.

1. Can you apply AA, SAS and SSS similarity theorems to decide whether a given triangles are similar or not?
2. Which of the following is (are) always correct?
   a. Congruent by SAS means similar by SAS.
   b. Similar by SAS means congruent by SAS.
   c. Congruent by SSS means similar by SSS.
   d. Similar by SSS means congruent by SSS.

? "But to decide whether two triangles are similar or not, it is necessary to know all the six facts stated in the definition 4.4?"

To prove similarity of triangles, using the definition of similarity means checking all the six conditions required by the definition. This is long and tiresome. Hence we want to have the minimum requirements which will guarantee us that the triangle are similar, i.e all the six conditions are satisfied. These short cut techniques are given as similarity theorems.

In this section you will see similarity theorems as you did see in grade 6 mathematics lessons congruence theorems for congruency of triangles.

Theorem 4.1: (AA-Similarity theorem)
If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.

Example 11: In Figure 4.28 below $\angle R \cong \angle V$ and show that $\Delta RSW \sim \Delta VSB$.

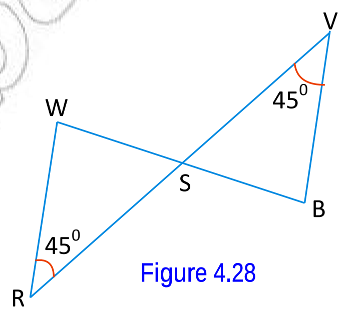


Figure 4.28

Proof:

Statements	Reasons
1. ∠R≅∠V	1. Degree measures are equal
2. ∠RSW≅∠VSB	2. Vertical opposite angles
3. ΔRSW~ΔVSB	3. AA similarity theorem

Example 12: In ΔABC and ΔXYZ if ∠ABC ≅ ∠XYZ, ∠ACB≅ ∠XZY, AB=8cm, AC=10cm and XY=4cm as shown in Figure 4.29 below; find the length of $\overline{XZ}$.

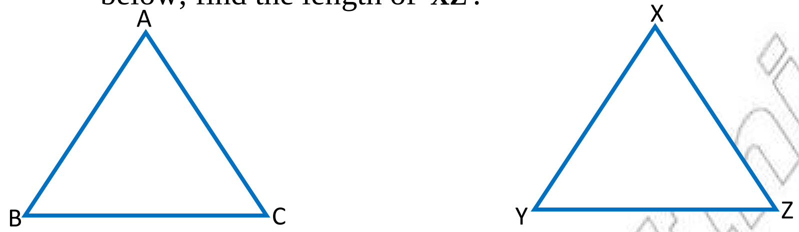


Figure 4.29

Solution: since ∠B ≅ ∠Y and ∠C ≅ ∠Z, we can say that

ΔABC ~ ΔXYZ by AA similarity theorem. Then,

$\frac{AB}{XY} = \frac{AC}{XZ}$ .... Definition of similar triangle.

Thus: $\frac{8}{4} = \frac{10}{XZ}$ ....Substitution

8XZ= 4 × 10 ....Cross multiplication

XZ= $\frac{40}{8} = 5\text{cm}$

Theorem 4.2: (SAS-Similarity theorem)

If two sides of one triangle are proportional to the corresponding two sides of another triangle and their included angles are also congruent, then the two triangles are similar.

Example 13: In Figure 4.30 below, find DE.

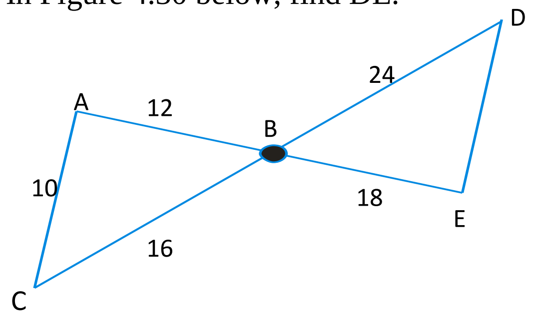


Figure 4.30

Solution:

1. $\angle ABC \cong \angle EBD$	1. Vertical opposite angles
2. $\frac{AB}{EB} = \frac{12}{18} = \frac{2}{3}$ and $\frac{BC}{BD} = \frac{16}{24} = \frac{2}{3}$	2. The ratio of the lengths of the corresponding sides are equal
3. $\Delta ABC \sim \Delta EBD$	3. SAS similarity theorem
4. $\frac{CA}{DE} = \frac{2}{3}$	4. Corresponding sides of similar triangles are proportional.
5. $\frac{10}{DE} = \frac{2}{3}$	5. Substitution
6. 2DE=30	6. Cross-product property
7. DE = 15 cm	7. solve for DE

Example 14: In Figure 4.31 below ΔABC and ΔDEF, are given where, $m(\angle A) = m(\angle D) = 55^0$, AB = 30 cm, AC = 100cm, DE = 15cm and DF = 50cm.

a. Are the two triangles similar?

b. Justify your answer.

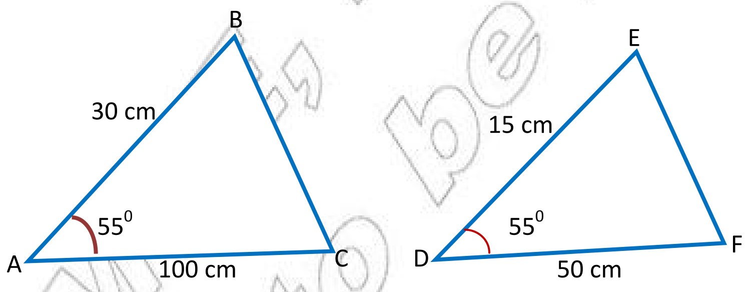


Figure 4.31

Solution:

a. Yes

b. Suppose ΔABC and ΔDEF are as shown in Figure 4.31 then,

$$\frac{AB}{DE}=\frac{30cm}{15cm}=2$$
$$\frac{AC}{DF}=\frac{100cm}{50cm}=2$$
$$\frac{AB}{DE}=\frac{AC}{DF}=2$$

Hence two sides of ΔABC are proportional to two corresponding sides of ΔDEF. Furthermore, m(∠A) = m(∠D) = 55^0, which shows that m(∠A) = m(∠D). Thus the included angles between the proportional sides of ΔABC and ΔDEF are congruent. Therefore ΔABC~ΔDEF by SAS similarity theorem.

Theorem 4.3: (SSS-Similarity theorem)
If the three sides of one triangle are in proportion to the three sides of another triangle, then the two triangles are similar.

Example 15: Based on the given Figure 4.32 below decide whether the two triangles are similar or not. Write the correspondence.

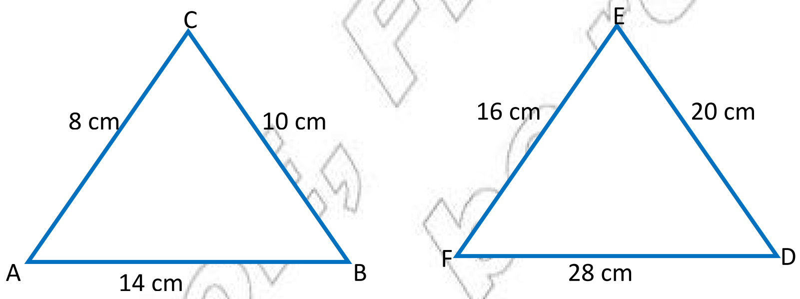


Figure 4.32

Solution:

$$\left.\begin{array}{l}\frac{AC}{FE}=\frac{8cm}{16cm}=\frac{1}{2}\\ \frac{CB}{ED}=\frac{10cm}{20cm}=\frac{1}{2}\\ \frac{AB}{FD}=\frac{14cm}{28cm}=\frac{1}{2}\end{array}\right\}$$

While finding proportional sides don't forget to compare the smallest with the smallest and the largest with the largest sides.

Hence $\frac{AC}{FE}=\frac{CB}{ED}=\frac{BA}{DF}=\frac{1}{2}$ or the sides are proportional.

Therefore $\Delta ABC \sim \Delta FDE$ .... By SSS similarity theorem.

From this you can conclude that: $\angle A \equiv \angle F$, $\angle B \equiv \angle D$ and $\angle C \equiv \angle E$.

Exercise 4D

1. If $\Delta ABC \sim \Delta XYZ$ and AC=10cm, AB=8cm and XY=4cm, find the length of $\overline{XZ}$.
2. Prove that any two equilateral triangles are similar.
3. In Figure 4.33 below determine the length x of the unknown side of ΔABC, if $\Delta ABC \sim \Delta DEF$.

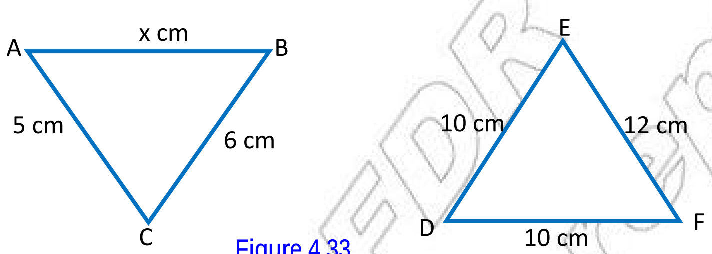


Figure 4.33

4. In Figure 4.34 below $\angle CAB \equiv \angle CDE$, AC=4cm, DC=5cm and DE=7cm. Determine the length of sides AB of ΔABC.

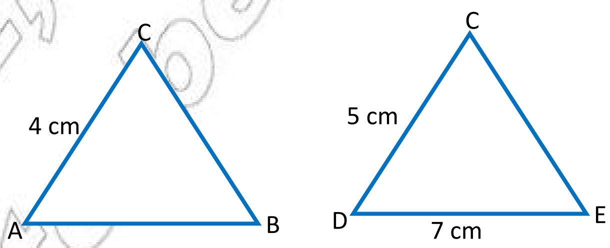


Figure 4.34

5. In Figure 4.35 of ΔABC, AC = 20cm, AB = 16cm, BC = 24cm.If D is a point on AC with CD = 15cm and, E is a point on BC with CE = 18cm, then:
   a. Show that ΔDEC~ΔABC.
   b. How long is DE?

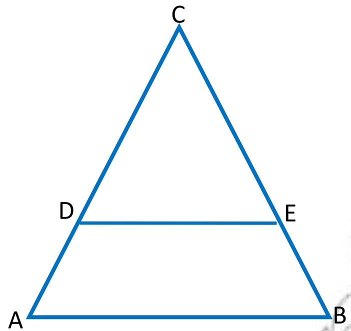


Figure 4.35

6. For any triangles ABC, if ∠A≡∠B, then show that ΔABC~ΔBAC.

7. Show that the corresponding altitudes of similar triangles ABC and PQR have the same ratio as two corresponding sides (See Figure 4.36).

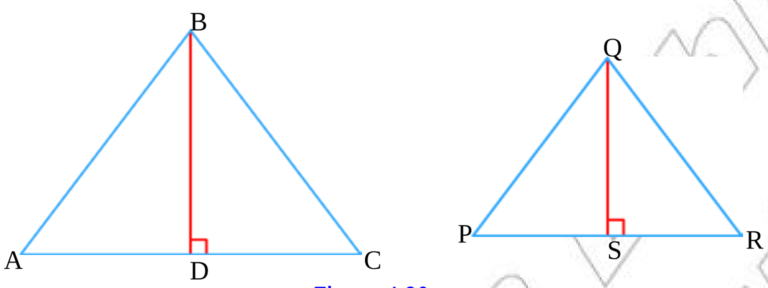


Figure 4.36

Challenge Problems

8. For the plane Figure 4.37 below $\overline{BE}$ and $\overline{AD}$ are altitude of ΔABC. prove that
   a. ΔADC~ΔBEC
   b. ΔAFE~ΔBFD

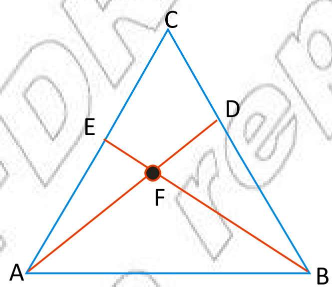


Figure 4.37

9. In Figure 4.38 to the right E is any point on $\overline{AB}$ and G is any point on $\overline{DC}$. If $\overline{AB}$// $\overline{DC}$, prove that:

   a. ΔDGF~ΔBEF. b. $\frac{DG}{EB}=\frac{DF}{FB}$.

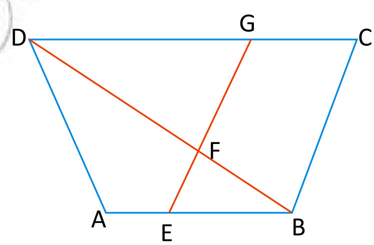


Figure. 4.38

10. In Figure 4.39 to the right determine the length AB of ΔABC.

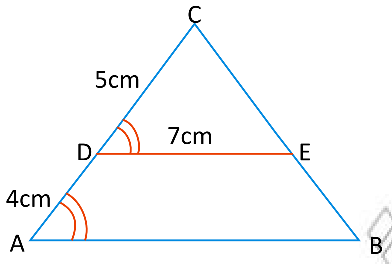


Figure 4.39

4.2.3 Perimeter and Area of Similar Triangles

Group work 4.3

1. Find the ratio of the areas of two similar triangles:

   a. if the ratio of their corresponding sides is $\frac{5}{4}$.

   b. if the ratio of their perimeters is $\frac{10}{9}$.

2. For the given Figure 4.40 below, find

   a. the area .

   b. the perimeter.

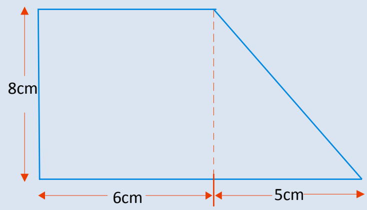


Figure 4.40

In lower grades you have seen how to find the perimeter and area of some special plane figures such as **triangles, rectangles, squares, parallelograms and trapeziums.** In the proceeding section of this unit you have been dealing with the areas and perimeters of similar plane figures. The perimeters and areas of similar plane figures have very interesting relations to their corresponding sides. You can compare the ratios of perimeters or that of the areas of similar polygons with out actually calculating the exact values of the perimeters or the areas. Look at the following example to help you clearly see these relations.

Example 16: In Figure 4.41 below if $\Delta ABC \sim \Delta XYZ$ with $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

Determine the relationship between:

a. The altitudes of the two triangles

b. The perimeters of the two triangles.

c. The areas of the triangles.

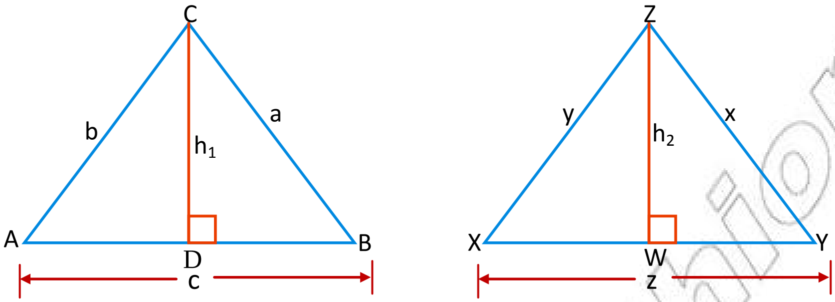


Figure. 4.41

Solution:

$\Delta ABC \sim \Delta XYZ$. let the constant of proportionality between their corresponding sides be K, i.e.

$\frac{a}{x} = k$ implies $a = kx$

$\frac{b}{y} = k$ implies $b=ky$

$\frac{c}{z} = k$ implies $c=kz$

$\frac{h_1}{h_2} = k$ implies $h_1=kh_2$

a. Let $\overline{CD}$ be the altitude of ΔABC from vertex C on $\overline{AB}$ and $\overline{ZW}$ be the altitude of ΔXYZ from vertex Z on $\overline{XY}$

Then $\angle CDB \cong \angle ZWY$ …. Both are right angles.

$\angle B \cong \angle Y$ …… Corresponding angles or similar triangles.

Therefore, $\Delta CDB \sim \Delta ZWY$ … By AA similarity theorem.

Thus $\frac{CD}{ZW} = \frac{CB}{ZY}$ ……. Definition of similar triangles.

$\Rightarrow \frac{h_1}{h_2} = \frac{a}{x}$ ……. Substitution

$\Rightarrow \frac{h_1}{h_2} = k$ ……. Since $\frac{a}{x} = k$ proportional sides

$\frac{h_1}{h_2} = k$ implies $h_1 = kh_2$

b. $P(\Delta ABC)=a+b+c$

$=kx+ky+kz$

$=k(x+y+z)$

and $p(\Delta xyz) = x+y+z$

then $\frac{P(\Delta ABC)}{P(\Delta XYZ)} = \frac{K(x+y+z)}{x+y+z} = k$

Hence the ratio of the perimeters of the two similar triangles is "k" which is equal to the ratio of the lengths of any pair of corresponding sides.

c. $a(\Delta ABC) = \frac{1}{2}\ c.h_1$

$= \frac{1}{2}(kzh_1)$

$= \frac{1}{2}(kz.kh_2)$

and $a(\Delta xyz) = \frac{1}{2}zh_2$

then $\frac{a(\Delta ABC)}{a(\Delta XYZ)} = \frac{\frac{1}{2}kz.kh_2}{\frac{1}{2}zh_2} = k^2$

Hence the **ratio of the areas of the two similar triangles** is k^2, the square of the ratio of the lengths of any pair of corresponding sides. The above examples will lead us to the following two important generalization which could be stated as theorems.

Theorem 4.4: If the ratios of the corresponding sides of two similar polygons is k, then the ratio of their perimeters is given by $\frac{P_1}{P_2} = \frac{S_1}{S_2} = k$.

Theorem 4.5: If the ratios of the corresponding sides of two similar polygons is $\frac{S_1}{S_2} = k$, then the ratio of their areas, is given by: $\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 = k^2$.

Example 17: Find the ratio of the areas of two similar triangles,

a. If the ratio of the corresponding sides is $\frac{5}{4}$.

b. If the ratio of their perimeters is $\frac{10}{9}$.

Solution:

Let A_1, A_2 be areas of two similar triangles, P_1, P_2 be the perimeters of the two triangles and S_1, S_2 be their corresponding sides.

a. $\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2$ ....Theorem 4.5

$\frac{A_1}{A_2} = \left(\frac{5}{4}\right)^2$ .... Substitution

Therefore, $\frac{A_1}{A_2} = \frac{25}{16}$.

b. $\frac{A_1}{A_2} = \left(\frac{P_1}{P_2}\right)^2$ .... Theorem 4.4 and 4.5

$\frac{A_1}{A_2} = \left(\frac{10}{9}\right)^2$ .... Substitution

Therefore, $\frac{A_1}{A_2} = \frac{100}{81}$.

Example 18: The areas of two similar polygons are $80cm^2$ and $5cm^2$. If a side of the smaller polygon is 2cm, find the corresponding sides of the larger polygons.

Solution:

Let A_1 and A_2 be areas of the two polygons and S_1, S_2 be their corresponding sides, then

$$\left(\frac{S_1}{S_1}\right)^2 = \frac{A_1}{A_2} \text{ .... Theorem 4.5}$$

$$\left(\frac{S_1}{S_1}\right)^2 = \frac{80}{5} \text{ .... Substitution}$$

$$\frac{S_1^2}{2^2} = \frac{80}{5}$$

$$\frac{S_1^2}{4} = 16$$

$$S_1^2 = 64$$

$$S_1 \times S_1 = 8 \times 8$$

$$S_1 = 8cm$$

Therefore, the corresponding sides of the larger polygon is 8cm.

Example 19: The sum of the perimeters of two similar polygon is 18cm. The ratios of the corresponding sides is 4:5. Find the perimeter of each polygon.

Solution:

Let S_1 and S_2 be the lengths of the corresponding sides of the polygon and P_1 and P_2 be their perimeters.

$P_1+P_2=18$

$P_1=18-P_2$ .... Solve for P_1

Thus $\frac{P_1}{P_2} = \frac{S_1}{S_2}$ ....Theorem 4.4

$\frac{18-P_2}{P_2} = \frac{4}{5}$ .... Substitution

$4P_2=90-5P_2$ .... Cross multiplication

$9P_2=90$

$P_2=10$ .... Divides both sides by 9.

Therefore, when $P_2=10$

$P_1+P_2=18$

$P_1+10=18$

$P_1=8$

Exercise 4E

1. In two similar triangles, find the ratio of:
   a. corresponding sides, if the areas are $50cm^2$ and $98cm^2$.
   b. the perimeter, if the areas are $50cm^2$ and $16cm^2$.
2. Two triangles are similar. The length of a side of one of the triangles is 6 times that of the corresponding sides of the other. Find the ratios of the perimeters and the area of the triangles.
3. The sides of a polygon have lengths 5, 7, 8, 11 and 19 cm. The perimeter of a similar polygon is 75cm. Find the lengths of the sides of larger polygon.
4. A side of a regular six – sided polygon is 8cm long. The perimeter of a similar polygon is 60cm. What is the length of a side of the larger polygon?
5. The ratio of the sides of two similar polygon is 3:2. The area of the smaller polygon is $24cm^2$. What is the area of the larger polygon?
6. Two trapeziums are similar. The area of one of the trapeziums is 4 times that of the other. Determine the ratios of the perimeters and the corres-ponding side lengths of the trapeziums.
7. Two triangles are similar. The length of a side of one of the triangles is 4 times that of the corresponding side of the other. Determine the ratios of the perimeters and the areas of the polygon.

Challenge Problems

8. Two pentagons are similar. The area of one of the pentagons is 9 times that of the other. Determine the ratios of the lengths of the corresponding sides and the perimeters of the pentagons.

9. Two triangles are similar. The length of a side of one of the triangles 2 times that of corresponding sides of the other. The area of the smaller triangle is 25sq.cm. Find the area of the larger triangle.
10. The lengths of the sides of a quadrilateral are 5cm, 6cm, 8cm and 11cm. The perimeter of a similar quadrilateral is 20cm. Find the lengths of the sides of the second quadrilateral.
11. The picture represents a man made pool surrounded by a park. The two quadrilateral are similar and the area of the pool is 1600 sq.cm. What is the area of the park of A'B' is four times the length of AB?

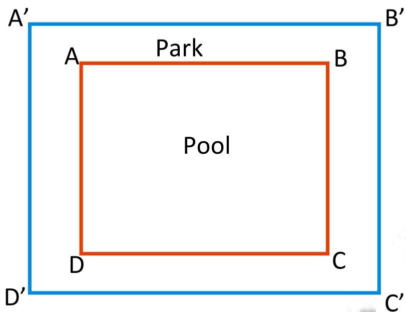


Figure 4.42

Summary For Unit 4

1. Similar geometric figures are figures which have the same shape.
2. Two polygons are similar if:
   i. their corresponding sides are proportional.
   ii. their corresponding angles are congruent.
3. Under an enlargement
   a. Lines and their images are parallel.
   b. Angles remain the same.
   c. All lengths are increased or decreased in the same ratio. .
4. **Scale factor-** the ratio of corresponding sides usually expressed numerically so that:

$$\text{scale factor} = \frac{\text{length of line on the enlargement}}{\text{length of line on the original}}$$

5. ΔABC similar to ΔDEF if:
   i. their corresponding sides are proportional.
   ii. their corresponding angles are congruent.
6. **AA Similarity theorems**

   If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.

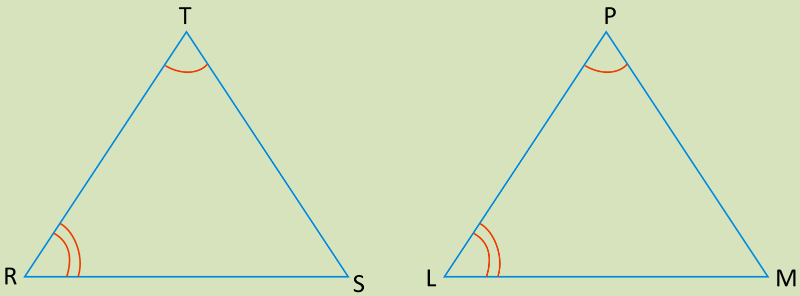


Figure 4.43

In the above Figure 4.43 you have:

- $\angle T \equiv \angle P$ if and only if $m(\angle T)=m(\angle P)$
- $\angle R \equiv \angle L$ if and only if $m(\angle R=m(\angle L)$
- $\Delta TRS \sim \Delta PLM$ .... by AA similarity.

7. ***SAS Similarity theorem***

If two sides of one triangle are proportional to the corresponding two sides of another triangle and their included angles are also congruent, then the two triangles are similar.

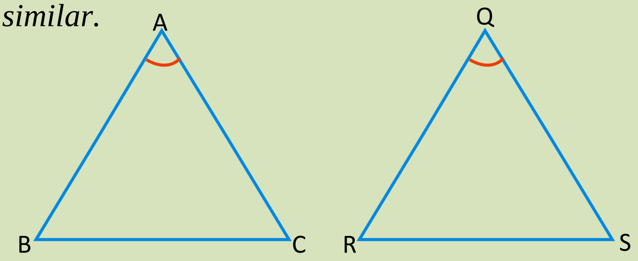


Figure 4.44

- $\angle A \equiv \angle Q$ *if and only if* $m(\angle A) = m(\angle Q)$
- $\dfrac{AB}{QR} = \dfrac{AC}{QS}$

Therefore $\Delta ABC \sim \Delta QRS$ .... by SAS similarity.

8. ***SSS Similarity theorem***

If the three sides of one triangle are in proportion to the three sides of another triangle, then the two triangles are similar.

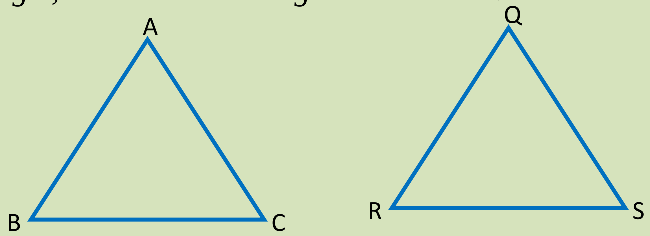


Figure 4.45

In the above Figure 4.45 you have:

$$\frac{AB}{QR} = \frac{BC}{RS} = \frac{AC}{QS}$$

Therefore, $\Delta ABC \sim \Delta QRS$ .... SSS Similarity.

9. In Figure 4.46, If $\Delta ABC \sim \Delta DEF$ with constant of proportionality k, then

a. $P(\Delta ABC)=KP(\Delta DEF)$

b. $a(\Delta ABC)=K^2 \times a(\Delta DEF)$

c. $AB = KDE$,

$BC = KEF$

$AC = KDE$

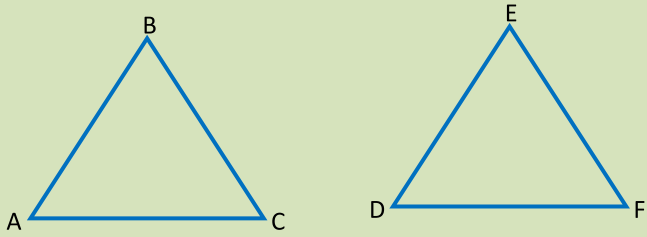


Figure 4.46

Miscellaneous Exercise 4

I Write true for the correct statements and false for the incorrect one.

1. $\overline{AB} \equiv \overline{CD}$ if and only if AB=CD.
2. ∠ABC≅∠DEF if and only if m(∠ABC)=m(∠DEF).
3. All rhombuses are similar.
4. All congruent polygons are similar.
5. All Isosceles triangles are similar.
6. Any two equilateral triangles are similar.

II Choose the correct answer from the given alternatives.

7. In Figure 4.47 given below $\overline{PQ} \perp \overline{QT}, \overline{ST} \perp \overline{QT}$ and P, R and S are on the same line. If ΔPQR~ΔSTR, then which of the following similarity theorems supports your answer?
   a. SAS Theorem
   b. AA Theorem
   c. SSS Theorem
   d. None

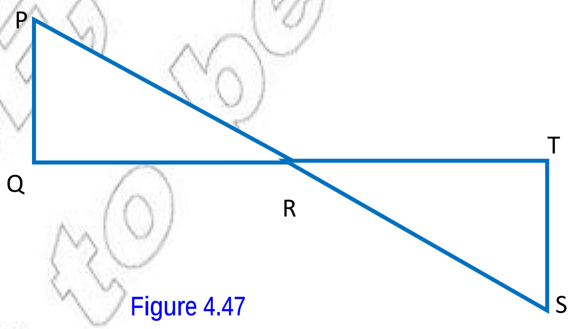


Figure 4.47

8. Given ΔABC and ΔDEF, if $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then which of the following postulates or theorem shows that ΔABC is similar to ΔDEF?

   a. AA similarity theorem.
   b. AAA similarity theorem.
   c. SAS similarity postulate.
   d. SSS similarity theorem.

9. Which of the following plane figures are not necessarily similar to each other?

   a. two equilateral triangles.
   b. two isosceles triangles.
   c. two circles.
   d. two squares.

10. Which of the following is different in meaning from ΔADF~ΔLMN?

    a. ΔDFA~ΔNML
    b. ΔFAD~ΔMNL
    c. ΔAFD~ΔLMN
    d. ΔDAF~ΔNLM

11. In Figure 4.48 below MN//YZ. If XN = 10cm, NZ = 5cm and MY = 4cm, then what is the length of $\overline{XY}$?

    a. 9cm
    b. 15 cm
    c. 12 cm
    d. 18 cm

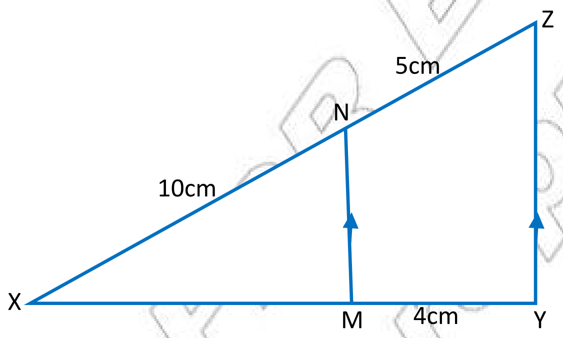


Figure 4.48

III Work out problems

12. Rectangle ABCD is similar to rectangle PQRS. Given that AB=14cm, BC=8cm and PQ=21 cm, calculate the length of QR.

13. A football field measures 100m by 72m. A school marks a football field similar in shape to a full size foot ball field but only 30m long. What is its width?

14. In Figure 4.49 below ∠QRP≅∠XYZ and show that ΔQRP~ΔXYZ.

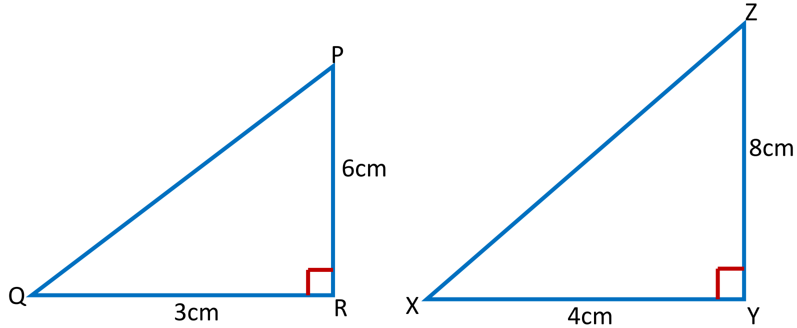


Figure 4.49

15. The sides of a polygon are 3cm, 5cm, 6cm, 8cm and 10cm. The perimeter of a similar polygons is 48cm. Find the sides of the second polygon.

16. In Figure 4.50 to the right $\overline{DC} \parallel \overline{AB}$, prove that $\frac{AO}{OD} = \frac{BO}{OC}$

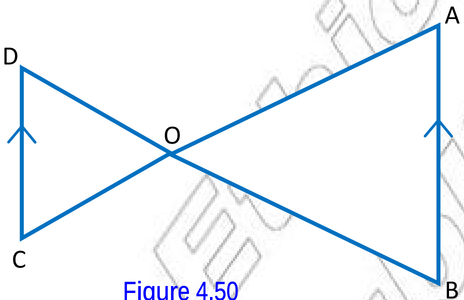


Figure 4.50

17. A piece of wood is cut as shown in Figure 4.51 below. The external and internal edge of the wood are similar quadrilaterals:

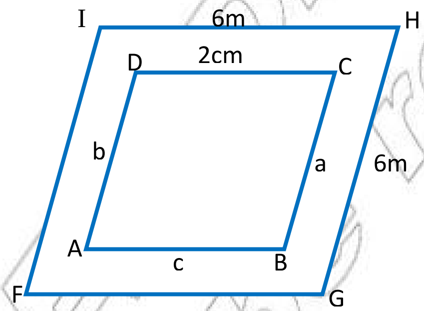


Figure 4.51

i.e. ABCD~ FGHI. The lengths of the sides are indicated on the figure. How long are the internal edges marked as a, b, and C?