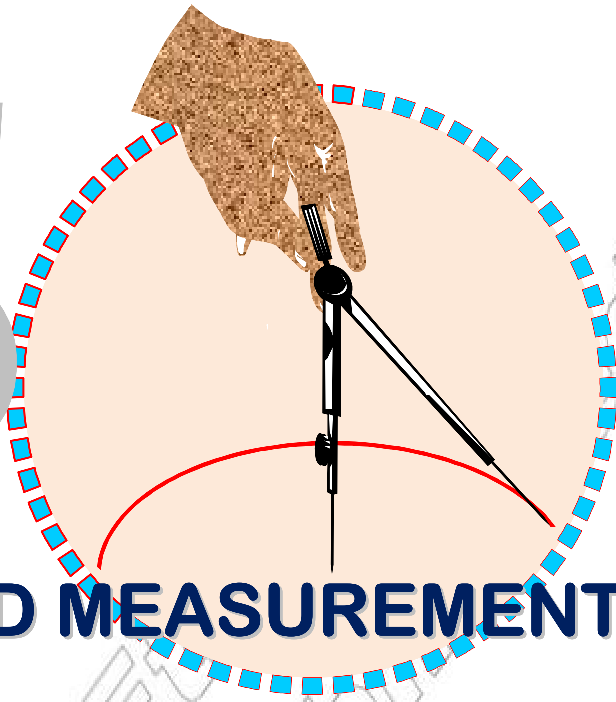
Unit 5



GEOMETRY AND MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about regular polygons.*
- *apply postulates and theorems in order to prove congruence and similarity of triangles.*
- *construct similar figures.*
- *apply the concept of trigonometric ratios to solve problems in practical situations;*
- *know specific facts about circles.*
- *solve problems on areas of triangles and parallelograms.*

Main Contents

INTRODUCTION

YOU HAVE LEARNT SEVERAL CONCEPTS, PRINCIPLES AND THEOREMS OF MEASUREMENT IN YOUR LOWER GRADES. IN THE P, YOU WILL LEARN MORE ABOUT GEOMETRY AND MEASUREMENT. REGULAR POLYGONS AND THEIR PROPERTIES, SIMILARITY OF TRIANGLES, RADIAN MEASURE OF AN ANGLE, TRIGONOM CIRCLES, PERIMETER AND AREA OF A SEGMENT AND A SECTOR OF A CIRC AND VOLUMES OF SOLIDS ARE THE MAJOR TOPICS COVERED IN THIS UNIT.

5.1 REGULAR POLYGONS

A Revision on polygons

THE FOLLOWING ACTIVITY MIGHT HELP YOU RECALL IMPORTANT FACTS ABOUT THAT YOU STUDIED IN PREVIOUS GRADES.

ACTIVITY 5.1

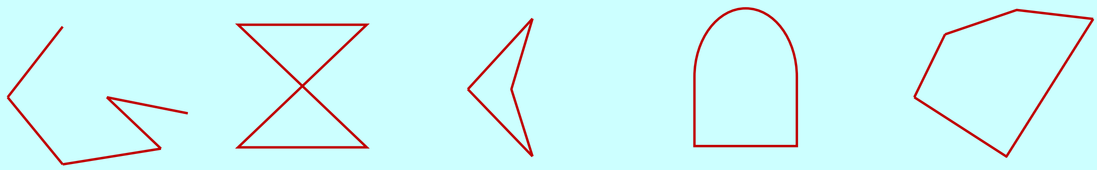
1 WHAT IS A POLYG

2 DISCUSS THE DIFFERENCE A CONVEX POLYGON AND A POLYGON.

3 FIND THE SUM OF THE MEASURES OF THE IN

A A TRIANGLE. B A QUADRILATERAL. C A PENTAG

4 WHICH OF THE FOLLOWIN ARE POLYGONS?



A B C D E

Figure 5.1

Definition 5.1

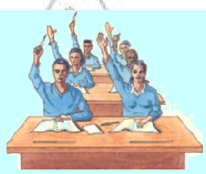
A polygon is a simple closed curve, formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

IN OTHER WORDS, A polygon IS A SIMPLE CLOSED PLANE SHAPE CONSISTING STRAIGHT-LINE SEGMENTS SUCH THAT NO TWO CONSECUTIVE LINE SEGMENTS ARE COLLINEAR.

B Interior and exterior angles of a polygon

WHEN REFERENCE IS MADE TO THE ANGLES OF A POLYGON, WE USUALLY M THE NAME INDICATES, AN interior angle IS AN ANGLE IN THE INTERIOR OF A POLY

ACTIVITY 5.2



1 DRAW A DIAGRAM TO HAT IS MEANT BY AN INTERIOR POLYGON.

2 A HOW MANY INTERIOR ANGL *n*-SIDED POLYGON H

B HOW MANY DIAGONALS FROM A VE *n*-SIDED POLYGON H

C INTO HOW MANY TRIANGL *n*-SIDED POLYGON BE PARTITIONED DIAGONALS FROM ONE

3 WHAT RELATIONS ARE TH BETWEEN THE NUMBER OF SIDES, THE NUM AND THE NUMBER OF INTERIOR ANG *n*-SIDED POLYGON

NOTE THAT THE NUMBER OF *vertices*, *angles* AND *sides* OF A POLYGON ARE T

Number of sides	Number of interior angles	Name of polygon
3	3	TRIANGLE
4	4	QUADRILATERAL
5	5	PENTAGON
6	6	HEXAGON
7	7	HEPTAGON
8	8	OCTAGON
9	9	NONAGON
10	10	DECAGON

Definition 5.2

An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called an exterior angle. It is formed between one side of the polygon and the extended adjacent side.

EXAMPLE 1 IN THE POLYGON ABCD IN FIGURE 5.2, ∠DCB IS AN INTERIOR ANGLE. ∠BCE AND ∠DCF ARE EXTERIOR ANGLES OF THE POLYGON AT THE VERTEX C.

(THERE ARE TWO POSSIBLE EXTERIOR ANGLES AT ANY VERTEX, WHICH ARE EQUAL.)

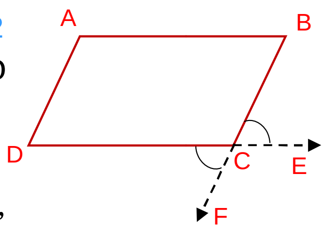


Figure 5.2

C The sum of the measures of the interior angles of a polygon

LET US FIRST CONSIDER THE SUM OF THE THE INTERIOR ANGLES OF A

ACTIVITY 5.3

- DRAW A FAIRLY LARGE TRIANGLE ON A SHEET

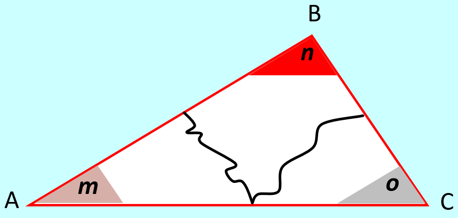


Figure 5.3

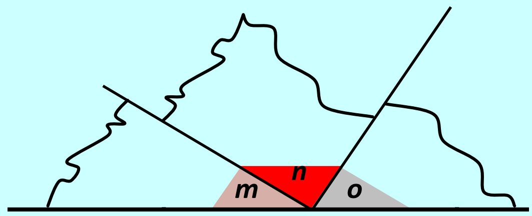


Figure 5.4

- NOW TEAR THE TRIANGLE INTO THREE PIECES, MAKING SURE E CORNER (AN
- FIT THESE THREE PIECES ALONG A STRAIGHT LINE IN FIGURE 5.4.

1 OBSERVE THAT THE SUM OF THE THREE ANGLES ANGLE.

2 WHAT IS THE SUM OF THE MEASU INTERIOR ANGLES OF ABC?

3 GIVEN BELOW ARE THE MEA ∠A, ∠B AND ∠C. CAN A TRIANGLE ABC BE WITH THE GIVEN ANGEXPLAIN.

A M (∠A) = 36^O; M (∠B) = 78^O; M (∠C) = 66^O.
B M (∠A) = 124^C ; M (∠B) = 56^O; M (∠C) = 20^O.
C M (∠A) = 90^O; M (∠B) = 74^O; M (∠C) = 18^O.

BASED ON OBSERVATIONS FROM ACTIVITY, WE STATE THE FOLLOWIN

Theorem 5.1 Angle sum theorem

The sum of the measures of the three interior angles of any triangle is 180^O.

ACTIVITY 5.4

PARTITIONING A POLYGON INTO TRIANIN FIGURE 5.6 CAN HE YOU TO DETERMINE THE SUM OF THE INTERIOR ANG

COMPLETE THE FOLLOWING

Number of sides of the polygon	Number of triangles	Sum of interior angles
3	1	1×180^{O}
4	2	2×180^{O}
5	3	3×180^{O}
6		$\_\_ \times 180^{O}$
7		
8		
n		$\_\_ \times 180^{O}$

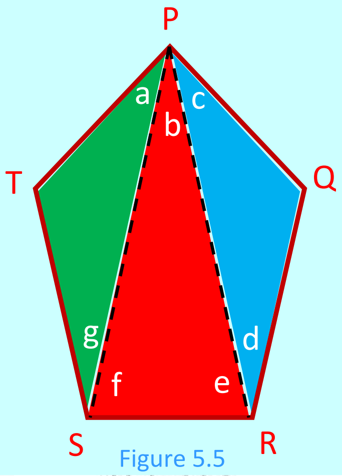


Figure 5.5

FROM THE ABOVE ACTIVITY, YOU CAN GENERALIZE THE SUM OF INTERIOR ANGLES OF A POLYGON AS FOLLOWS:

Theorem 5.2

If the number of sides of a polygon is n, then the sum of the measures of all its interior angles is equal to $(n - 2) \times 180^{O}$.

FROM ACTIVITY 5.4 AND THEOREM 5.2, YOU CAN ALSO OBSERVE THAT AN n-SIDED POLYGON CAN BE DIVIDED INTO $(n - 2)$ TRIANGLES. SINCE THE SUM OF INTERIOR ANGLES OF A TRIANGLE IS 180^{o}, THE SUM OF THE ANGLES OF $(n - 2)$ TRIANGLES IS GIVEN BY:

$$S = (n - 2) \times 180^{o}.$$

ACTIVITY 5.5

1 USING FIGURE 5.6, VERIFY THE FORMULA $S = (n - 2) \times 180^{o}$ GIVEN ABOVE.

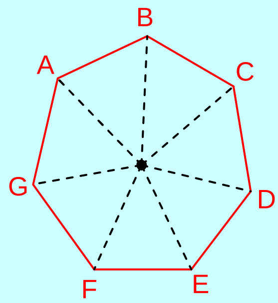
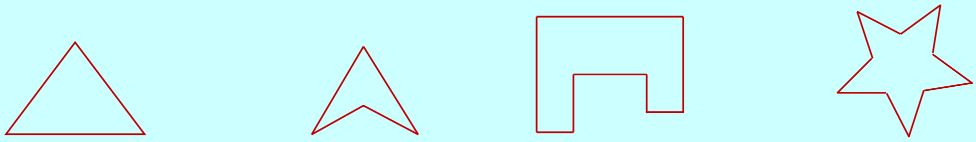


Figure 5.6

Hint: ANGLES AT A POINT ADD UP TO 360^{O}.

2 BY DIVIDING EACH OF THE FOLLOWING FIGURES INTO TRIANGLES, SHOW THAT THE FORMULA $S = (n - 2) \times 180^o$ FOR THE SUM OF THE MEASURES OF ALL INTERIOR ANGLES OF AN n-SIDED POLYGON IS VALID FOR EACH OF THE FOLLOWING POLYGONS:



A **B** **C** **D**

Figure 5.7

3 IN A QUADRILATERAL ABCD, IF m(∠A) = 80^o, m(∠B) = 100^o AND m(∠D) = 110^o, FIND m(∠C).

4 IF THE MEASURES OF THE INTERIOR ANGLES OF A HEXAGON ARE x^O, $2x^O$, 60^O, $(x + 30)^O$, $(x - 10)^O$ AND $(x + 40)^O$, FIND THE VALUE OF x.

5 **A** LET i_1, i_2, i_3 BE THE MEASURES OF THE INTERIOR ANGLES OF THE GIVEN TRIAN
LET e_1, e_2 AND e_3 BE THE MEASURES OF THE EXTERIOR ANGLES, AS INDICA
IN FIGURE 5.8.

EXPLAIN EACH STEP IN THE FOLLOWING:

$i_1 + e_1 = 180^O$

$i_2 + e_2 = 180^O$

$i_3 + e_3 = 180^O$

$(i_1 + e_1) + (i_2 + e_2) + (i_3 + e_3) = 180^o + 180^o + 180^o$

$(i_1 + i_2 + i_3) + (e_1 + e_2 + e_3) = 3 \times 180^o$

$180^o + e_1 + e_2 + e_3 = 3 \times 180^o$

$e_1 + e_2 + e_3 = 3 \times 180^o - 180^o = 2 \times 180^o$

$e_1 + e_2 + e_3 = 360^o$

THAT IS, *the sum of the measures of the exterior angles of a triangle, taking one angle at each vertex, is* 360^o.

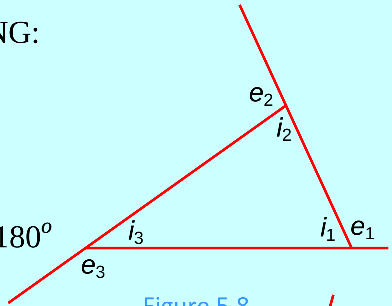


Figure 5.8

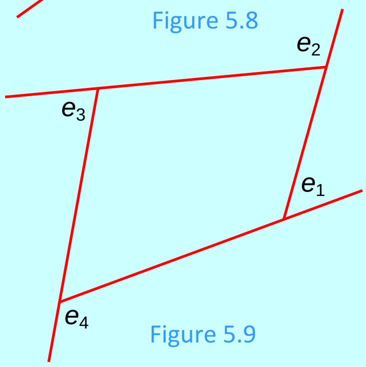


Figure 5.9

B REPEAT THIS FOR THE QUADRILATERAL GIVEN IN FIGURE 5.9. FIND THE SUM OF THE MEASURES OF THE EXTERIOR ANGLES OF THE QUADRILATERAL. I.E., FIND $e_1 + e_2 + e_3 + e_4$.

C IF $e_1, e_2, e_3 \ldots e_n$ ARE THE MEASURES OF THE EXTERIOR ANGLES OF AN n-SIDED POLYGON, THEN $e_1 + e_2 + e_3 + \ldots + e_n$ = ______.

6 SHOW THAT THE MEASURE OF AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO T
MEASURES OF THE TWO OPPOSITE INTERIOR ANGLES.

5.1.1 Measures of Angles of a Regular Polygon

SUPPOSE WE CONSIDER A CIRCLE WITH CENTRE O AND RADIUS r, AND DIVIDE THE CIRCLE INTO n EQUAL ARCS. (THE FIGURE GIVEN ON THE RIGHT SHOWS THIS WHEN $n = 8$.)

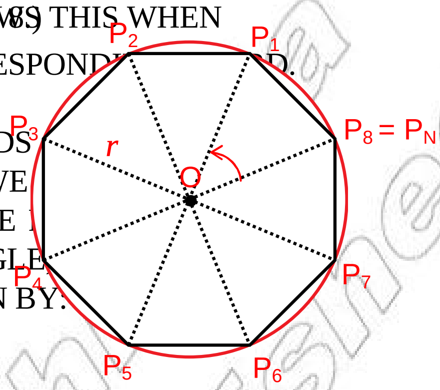


Figure 5.10

FOR EACH LITTLE ARC, WE DRAW THE CORRESPOND... D. THIS GIVES A POLYGON WITH VERTICES $P_1, P_2, \ldots, P_n$. SINCE THE ARCS HAVE EQUAL LENGTHS, THE CHORDS ... SIDES OF THE POLYGON) ARE EQUAL. IF WE ... SEGMENTS FROM O TO EACH VERTEX OF THE ... GET n ISOSCELES TRIANGLES. IN EACH TRIANGLE ... MEASURE OF THE CENTRAL ANGLE O IS GIVEN BY:

$$m(\angle O) = \frac{360^o}{n}.$$

SINCE THE VERTEX ANGLES AT O OF EACH ISOSCELES TRIANGLE HAVE EQUAL MEAS $\frac{360^o}{n}$, IT FOLLOWS THAT ALL THE BASE ANGLES OF ALL THE ISOSCELES TRIANGLES ARE

FROM THIS, IT FOLLOWS THAT THE MEASURES OF ALL THE ANGLES OF THE POLYGON MEASURE OF AN ANGLE OF THE POLYGON IS TWICE THE MEASURE OF ANY BASE ANGLE THE ISOSCELES TRIANGLES. SO, THE POLYGON HAS ALL OF ITS SIDES EQUAL AND AL EQUAL. A POLYGON OF THIS TYPE IS CALLED A **regular polygon**.

Definition 5.3

A **regular polygon** is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angles are equal.

NOTE THAT THE MEASURE OF AN INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{S}{n}$, WHERE $S = (n-2) \times 180^{\text{O}}$ IS THE SUM OF THE MEASURES OF ALL OF ITS INTERIOR ANGLES. HENCE, WE HAVE THE FOLLOWING:

THE MEASURE OF EACH INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{(n-2)180^o}{n}$.

A POLYGON IS SAID TO BE INSCRIBED IN A CIRCLE IF ALL OF ITS VERTICES LIE ON THE CIRC

FOR EXAMPLE, THE QUADRILATERAL SHOWN IN FIGURE 5.11 IS INSCRIBED IN THE CIRCLE.

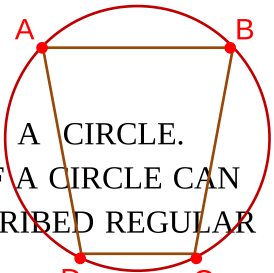


Figure 5.11

ANY REGULAR POLYGON CAN BE INSCRIBED IN A CIRCLE. BECAUSE OF THIS, THE CENTRE AND THE RADIUS OF A CIRCLE CAN BE TAKEN AS THE CENTRE AND RADIUS OF AN INSCRIBED REGULAR POLYGON.

EXAMPLE 1

I FIND THE MEASURE OF EACH INTERIOR ANGLE AND EACH CENTRAL ANGLE OF A REGULAR POLYGON WITH:

A 3 SIDES **B** 5 SIDES

II FIND THE MEASURE OF EACH EXTERIOR ANGLE OF A REGULAR n-SIDED POLYGON.

SOLUTION:

I **A** SINCE THE SUM OF INTERIOR ANGLES OF A TRIANGLE IS 180^o, EACH INTERIOR ANGLE IS $\frac{180^o}{3} = 60^o$.

RECALL THAT A 3-SIDED REGULAR POLYGON IS AN equilateral triangle.

TO FIND THE MEASURE OF A CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON, RECALL THAT the sum of the measures of angles at a point is 360°. HENCE, THE SUM OF THE MEASURES OF THE CENTRAL ANGLES IS 360° (FIGURE 5.11 ILLUSTRATES THIS FOR $n = 8$). SO,

THE MEASURE OF EACH CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$. FROM

THIS, WE CONCLUDE THAT THE MEASURE OF EACH CENTRAL ANGLE OF AN EQUIL

IS $\frac{360^o}{3} = 120^o$.

B RECALL THAT THE SUM OF ALL INTERIOR ANGLES OF A 5-SIDED POLYGON IS $(5 - 2) \times 180^o = 3 \times 180^o = 540^o$. SO, THE MEASURE OF EACH interior angle OF A regular pentagon IS $540^o \div 5 = 108^o$.

ALSO, THE MEASURE OF EACH central angle OF A regular pentagon IS $\frac{360^o}{5} = 72^O$.

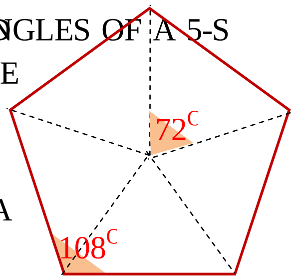


Figure 5.12

II TO FIND THE MEASURE OF EACH EXTERIOR ANGLE IN A REGULAR n-SIDED POLYGON, NOTICE THAT AT EACH VERTEX, THE SUM OF AN INTERIOR ANGLE AND AN EXTERIOR ANGLE IS 180°.

(*See* FIGURE 5.13).

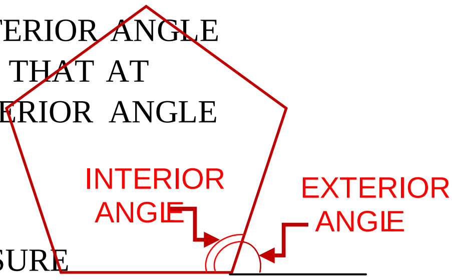


Figure 5.13

HENCE EACH EXTERIOR ANGLE WILL MEASURE

$$180^o - \left[\frac{(n-2)}{n} 180^o\right] = \frac{n180^o - (n-2)180^o}{n} = \frac{360^o}{n},$$

WHICH IS THE SAME AS THE MEASURE OF A CENTRAL ANGLE.

WE CAN SUMMARIZE OUR RESULTS ABOUT ANGLE MEASURES IN REGULAR POLYGONS A

FOR ANY REGULAR n-SIDED POLYGON:

I MEASURE OF EACH INTERIOR ANGLE = $\dfrac{(n-2)180^o}{n}$

II MEASURE OF EACH CENTRAL ANGLE = $\dfrac{360^o}{n}$

III MEASURE OF EACH EXTERIOR ANGLE = $\dfrac{360^o}{n}$

Exercise 5.1

1 IN FIGURE 5.14A, NO TWO LINE SEGMENTS THAT ARE IN SUCCESSION ARE COLLINEAR AND NO TWO SEGMENTS INTERSECT EXCEPT AT THEIR END POINTS. YET THE FIGURE IS NO WHY NOT?

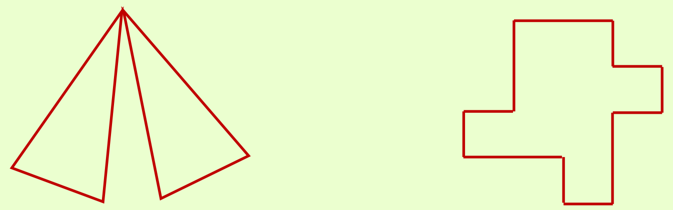


Figure 5.14

2 IS FIGURE 5.14B A POLYGON? HOW MANY SIDES DOES IT HAVE? HOW MANY VERTI DOES IT HAVE? WHAT IS THE SUM OF THE MEASURES OF ALL OF ITS INTERIOR ANG

3 ABCD IS A QUADRILATERAL SUCH THAT THE MEASURES OF THREE OF ITS INTERIOR ANGLES ARE GIVEN AS m(∠D) = 112^o, m(∠C) = 75^o AND m(∠B) = 51^o. FIND m(∠A).

4 FIND THE MEASURE OF AN INTERIOR ANGLE OF A REGULAR POLYGON WITH:

A 10 SIDES **B** 20 SIDES **C** 12 SIDES

5 FIND THE NUMBER OF SIDES OF A REGULAR POLYGON, IF THE MEASURE OF EACH OF ITS INTERIOR ANGLES IS:

A 150^o **B** 160^o **C** $147.\overline{27}^o$

6 IF THE MEASURE OF A CENTRAL ANGLE OF A REGULAR POLYGON IS 18^o, FIND THE MEASURE OF EACH OF ITS INTERIOR ANGLES.

7 **I** CAN A REGULAR POLYGON BE DRAWN SUCH THAT THE MEASURE OF EACH EXTERIOR ANGLE IS:

A 20^o? **B** 16^o? **C** 15^o?

IN EACH CASE, IF YOUR ANSWER IS NO, JUSTIFY IT; IF YES, FIND THE NUMBER OF SI

II CAN A REGULAR POLYGON BE DRAWN SUCH THAT THE MEASUR:

A 144^o? **B** 140^o? **C** 130^o?

IN EACH CASE IF YOUR ANS', FIND THE NUMBER OF SID JUSTIFY IT.

8 ABCDEFGH IS A REGULAR OCTAGON $\overline{AB}$ AND $\overline{DC}$ ARE PRODUCED TO MI FIND M(∠AND).

9 FIGURE 5.15 REPRESENTS PART OF A REGULAR POL $\overline{AB}$ ANI $\overline{BC}$ ARE SIDES, AND R IS THE CEOF THE CIRCLE IN WHICH THE POLYGON IS INSCI COMPLETE THE FOLLOW:

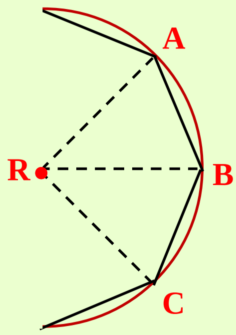


Figure 5.15

Number of sides	m(∠ARB) or m(∠BRC)	m(∠ABR) or m(∠CBR)	m (∠ABC)
3			
4			
5			
6			
	45^C		
9	40^C	70^C	140^C
			144^C
12			
15			
18			
20			

5.1.2 Properties of Regular Polygons

ACTIVITY 5.6

1 WHICH OF THE FOLLOWING PLANE FIGURES CAN BE D TWO IDENTICAL PARTS BY DRAWING A LINI (IN OTHI WORDS, WHICH OF THE FOLLOWING PLANE FIGURES SYMMETRY?) DIS(

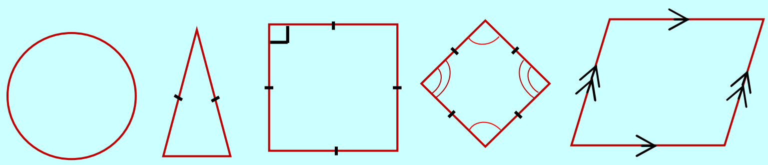


Figure 5.16

2 WHICH OF THE ABOVE FIGURES HAVE MORE THAN ONE

3 HOW MANY LINES OF SYI DOES A REGU SIDED POLYGON HAVE? n-

A FIGURE HAS A line of symmetry, IF IT CAN BE FOLDED SO THAT ONE HALF OF THE FIGURE COINCIDES WITH
THE ETHIOPIAN FLAG HAS A LINE OF SYMMETRY
BROKEN LINE SHOWN. THE RIGHT HALF IS A REFL
HALF, AND THE CENTRE LINE IS THE LINE OF REFL

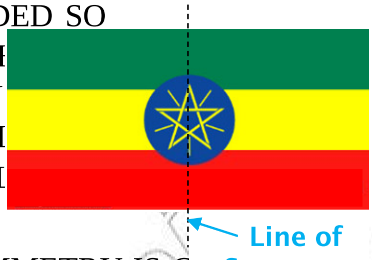


Figure 5.17

THE LINE OF REFLECTION IS ALSO CALLED THE line of symmetry.

A FIGURE THAT HAS AT LEAST ONE LINE OF SYMMETRY IS C symmetrical figure.

SOME FIGURES HAVE MORE THAN ONE LINE OF SYMMETRY. IN SUCH CASES, THE SYMMETRY ALWAYS INTERSECT AT ONE POINT.

NOTE THAT EQUILATERAL TRIANGLES, SQUARES AND REGULAR PENTAGONS HAVE A SYMMETRY AS THEIR SIDES.

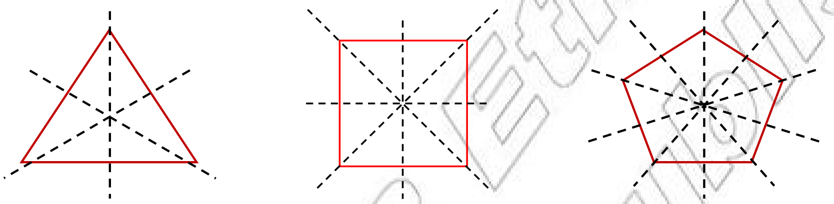


Figure 5.18

TO GENERALIZE, A REGULAR n-SIDED POLYGON ALWAYS HAS n LINES OF SYMMETRY.

Circumscribed regular polygons

A POLYGON WHOSE SIDES ARE TANGENT TO A CIRCLE IS SAID TO circumscribe the circle. FOR EXAMPLE, THE QUADRILATERAL KLMN *circumscribes* THE CIRCLE. THE CIRCLE IS *inscribed* IN THE QUADRILATERAL.

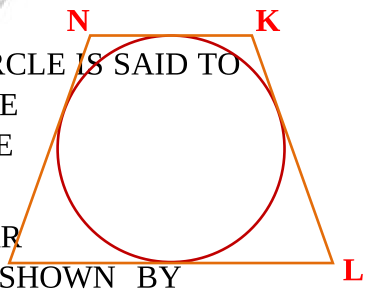


Figure 5.19

IT IS POSSIBLE TO CIRCUMSCRIBE ANY n-SIDED REGULAR POLYGON ABOUT A CIRCLE. THE METHOD IS SHOWN BY CIRCUMSCRIBING A 5-SIDED POLYGON, IN FIGURE 5.20

THE IDEA IS THAT THE RADII TO THE POINTS OF TANGENCY MAKE 5 CONGRUENT ANGLES AT THE CENTRE WHOSE MEASURES ADD UP TO 360°. USING A PROTRACTOR, WE CONSTRUCT FIVE RADII, MAKING ADJACENT CENTRAL ANGLES OF $\frac{360^o}{5} = 72^o$. THE RADII END AT S, R, W, X, Y.

LINE SEGMENTS PERPENDICULAR TO THE RADII AT THEIR ENDPOINTS ARE TANGENT TO THE CIRCLE AND FORM THE CIRCUMSCRIBED PENTAGON AS DESIRED.

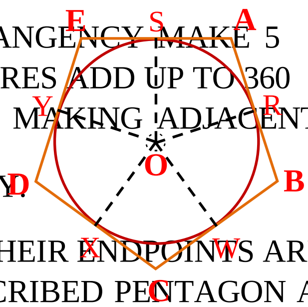


Figure 5.20

Can you see how the vertices A, B, C, D, E are determined?

REGULAR POLYGONS HAVE A SPECIAL RELATION TO CIRCLES. A REGULAR POLYGON INSCRIBED IN OR CIRCUMSCRIBED ABOUT A CIRCLE.

THIS LEADS US TO STATE THE FOLLOWING PROPERTY ABOUT REGULAR POLYGONS:

A CIRCLE CAN ALWAYS BE INSCRIBED IN OR CIRCUMSCRIBED ABOUT ANY GIVEN REGUL

IN FIGURE 5.20 ABOVE, THE RADIUS OX OF THE INSCRIBED CIRCLE IS THE DISTANCE F CENTRE TO THE SIDE (CD) OF THE REGULAR POLYGON. THIS DISTANCE FROM THE CENT OF THE POLYGON, DENOTED BY a, IS THE SAME. THIS DISTANCE a IS CALLED THE apothem OF THE REGULAR POLYGON.

Definition 5.4

The distance a from the centre of a regular polygon to a side of the polygon is called the apothem of the polygon. That is, the apothem a of a regular polygon is the length of the line segment drawn from the centre of the polygon perpendicular to the side of the polygon.

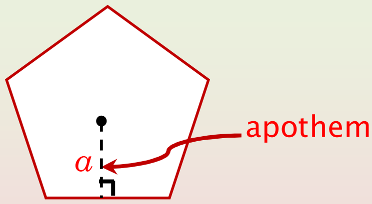


Figure 5.21

Regular Pentagon

THE FOLLOWING EXAMPLE ILLUSTRATES HOW TO FIND PERIMETER, AREA AND APOTHE

EXAMPLE 1 IN FIGURE 5.22, THE REGULAR PENTAGON ABCDE IS INSCRIBED IN A CIRCLE WITH CENTRE O AND RADIUS r. WRITE FORMULAE FOR THE SIDE s, PERIMETER P, APOTHEM a AND AREA A OF THE REGULAR PENTAGON.

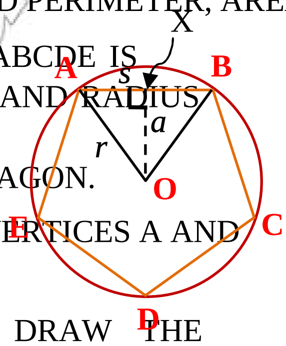


Figure 5.22

SOLUTION: TO SOLVE THE PROBLEM, JOIN O TO THE VERTICES A AND B AS SHOWN SO THAT ΔOAB IS FORMED.

ΔOAB IS AN ISOSCELES TRIANGLE (WHY?). DRAW THE PERPENDICULAR FROM O TO $\overline{AB}$. IT MEETS $\overline{AB}$ AT X. ∠AOB IS A CENTRAL ANGLE OF THE REGULAR PENTAGON.

SO, M (∠AOB) = $\frac{360^o}{5} = 72^o$.

NOW ΔAOX ≅ ΔBOX (*verify this*).

THEREFORE, ∠AOX ≅ ∠BOX.

THEREFORE, M(∠AOX) = M(∠BOX) = $\frac{1}{2}$ M(∠AOB) = $\frac{1}{2}(72^o) = 36^o$.

LET s = AB, THE LENGTH OF THE SIDE OF THE REGULAR PENTAGON.

SINCE ΔAOX ≅ ΔBOX, WE HAVE $\overline{AX} \cong \overline{XB}$. SO, AX = $\frac{1}{2}$ AB = $\frac{1}{2}s$.

NOW IN THE RIGHT ANGLED TRIANGLE AOX YOU SEE THAT

$$\text{SIN}(\angle AOX) = \frac{AX}{AO}\text{. I.E., SIN}\left(\frac{1}{2}(\angle AOB)\right) = \frac{\frac{1}{2}s}{r}$$

$$\text{SIN } 36^o = \frac{\frac{1}{2}s}{r}\text{. SO } \frac{1}{2}s = r \text{ SIN } 36^o$$

THEREFORE, $s = 2r \sin 36^{O}$ . . . . (1)

PERIMETER P OF THE POLYGON IS

P = AB + BC + CD + DE + EA

BUT SINCE AB = BC = CD = DE = EA = s, WE HAVE $P = s + s + s + s + s = 5s$.

SINCE FROM (1) WE HAVE $s = 2r$ SIN 36^o, THE PERIMETER OF THE REGULAR PENTAGON IS

$P = 5 \times 2r \sin 36^{O}$

$\therefore$ $P = 10r \sin 36^{O}$ . . . . (2)

TO FIND A FORMULA FOR THE APOTHEM a, CONSIDER ΔAOX

$$\text{COS}(\angle AOX) = \frac{XO}{AO}$$

SINCE M($\angle$AOX) = 36^o, XO = a, AND AO = r.

$$\text{COS}(36^o) = \frac{a}{r}$$

SO, $a = r \cos 36^{O}$ . . . . (3)

TO FIND THE AREA OF THE REGULAR PENTAGON, FIRST WE FIND THE AREA OF ΔAOB. TAKING THE HEIGHT AND THE BASE OF ΔAOB AS OX AND AB, RESPECTIVELY, WE HAVE,

$$\text{AREA OF } \Delta AOB = \frac{1}{2} AB \times OX = \frac{1}{2} \times s \times a = \frac{1}{2} as$$

NOW THE AREA OF THE REGULAR PENTAGON ABCDE = AREA OF ΔAOB + AREA OF ΔBOC + AREA OF ΔCOD + AREA OF ΔDOE + AREA OF ΔEOA.

SINCE ALL THESE TRIANGLES ARE CONGRUENT, THE AREA OF EACH TRIANGLE IS $\frac{1}{2}as$.

SO, THE AREA OF THE REGULAR PENTAGON ABCDE $= 5 \times \left(\frac{1}{2}as\right) = \frac{1}{2}a(5s) = \frac{1}{2}aP$ . . . (4)

SINCE $36^o = \frac{180^o}{5}$, WHERE 5 IS THE NUMBER OF SIDES, WE CAN GENERALIZE THE AI

FORMULAE FOR n-SIDED REGULAR POLYGON BY REPLACING 36^o BY $\frac{180^o}{n}$, AS FOLLOWS.

Theorem 5.3

Formulae for the length of side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are

1 $s = 2r\sin\frac{180^o}{n}$

2 $a = r\cos\frac{180^o}{n}$

3 $P = 2nr\sin\frac{180^o}{n}$

4 $A = \frac{1}{2}aP$

EXAMPLE 2

A FIND THE LENGTH OF THE SIDE OF AN EQUILATERAL TRIANGLE WHOSE RADIUS IS $\sqrt{12}$ CM.

B FIND THE AREA OF A REGULAR HEXAGON WHOSE RADIUS IS 5 CM.

C FIND THE APOTHEM OF A SQUARE WHOSE RADIUS IS $\sqrt{8}$ CM.

SOLUTION:

A BY THE FORMULA, THE LENGTH OF THE SIDE IS $s = 2r\sin\frac{180^o}{n}$.

SO, REPLACING r BY $\sqrt{12}$ AND n BY 3, WE HAVE,

$$s = 2\times\sqrt{12}\times\sin\frac{180^o}{3} = 2\times\sqrt{12}\times\sin 60^o$$

$$= 2\times\sqrt{12}\times\frac{\sqrt{3}}{2} = \sqrt{12\times3} = \sqrt{36} = 6;\quad \left(\sin 60^o = \frac{\sqrt{3}}{2}\right)$$

THEREFORE, THE LENGTH OF THE SIDE OF THE EQUILATERAL TRIANGLE IS 6 CM.

B TO FIND THE AREA OF THE REGULAR HEXAGON, WE USE THE FORMULA

$A = \frac{1}{2}aP$, WHERE a IS THE APOTHEM AND P IS THE PERIMETER OF THE REGULAR HEXAGON.

THEREFORE,

$$A = \frac{1}{2}aP = \frac{1}{2}\left(r\cos\frac{180^o}{n}\right)\left(2nr\sin\frac{180^o}{n}\right) \quad \textit{(Substituting formulae for a and P)}$$

$$= \frac{1}{2}\times\left(5\times\cos\frac{180^o}{6}\right)\times\left(2\times6\times5\sin\frac{180^o}{6}\right)$$

$$= \frac{1}{2}\times5\times\frac{\sqrt{3}}{2}\times2\times6\times5\times\frac{1}{2};\quad \left(\cos 30^o = \frac{\sqrt{3}}{2},\ \sin 30^o = \frac{1}{2}\right)$$

$$= \frac{75\sqrt{3}}{2}\ \text{CM}^2$$

c TO FIND THE APOTHEM OF THE SQUARE, WE USE THE FORMULA $a = r\cos\frac{180^o}{n}$.

REPLACING r BY $\sqrt{8}$ AND n BY 4, WE HAVE

$$a = \sqrt{8}\cos\frac{180^o}{4} = \sqrt{8}\cos 45^o \qquad \left(\cos 45^o = \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{8}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{16}}{2} = 2 \text{ CM.}$$

Exercise 5.2

1. WHICH OF THE CAPITAL LETTERS OF THE ENGLISH ALPHABET ARE LINE SYMMETRICAL?
2. DRAW ALL THE LINES OF SYMMETRY ON A DIAGRAM OF A REGULAR:

   a HEXAGON **b** HEPTAGON **c** OCTAGON

   HOW MANY LINES OF SYMMETRY DOES EACH ONE HAVE?
3. IF A REGULAR POLYGON OF n SIDES HAS EVERY LINE OF SYMMETRY PASSING THROUGH A VERTEX, WHAT CAN YOU SAY ABOUT n?
4. STATE WHICH OF THE FOLLOWING STATEMENTS ARE TRUE AND WHICH ARE FALSE:

   a A PARALLELOGRAM WHICH HAS A LINE OF SYMMETRY IS A RECTANGLE.

   b A RHOMBUS WHICH HAS A LINE OF SYMMETRY MUST BE A SQUARE.

   c AN ISOSCELES TRIANGLE WITH MORE THAN ONE LINE OF SYMMETRY IS AN EQUILATERAL TRIANGLE.

   d A PENTAGON THAT HAS MORE THAN ONE LINE OF SYMMETRY MUST BE REGULAR.
5. SHOW THAT THE LENGTH OF EACH SIDE OF A REGULAR HEXAGON IS EQUAL TO THE LENGTH OF THE RADIUS OF THE HEXAGON.
6. SHOW THAT THE AREA A OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = 2r^2$.
7. DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE:

   a THE AREA OF AN EQUILATERAL TRIANGLE WITH APOTHEM $\sqrt{3}$ CM AND SIDE 6 CM IS $9\sqrt{3}$ CM^2.

   b THE AREA OF A SQUARE WITH APOTHEM $\sqrt{2}$ CM AND SIDE $2\sqrt{2}$ CM IS $8\sqrt{2}$ CM^2.
8. FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR NINE-SIDED POLYGON WITH RADIUS 5 UNITS.

9 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR TWELVE-SIDED POLYGON WITH RADIUS 3 CM.

10 FIND THE RATIO OF THE PERIMETER OF A REGULAR HEXAGON TO ITS RADIUS AND SHOW THAT THE RATIO DOES NOT DEPEND ON THE RADIUS.

11 FIND THE RADIUS OF AN EQUILATERAL TRIANGLE WITH PERIMETER 24 UNITS.

12 FIND THE RADIUS OF A SQUARE WITH PERIMETER 32 UNITS

13 FIND THE RADIUS OF A REGULAR HEXAGON WITH PERIMETER 48 UNITS.

14 THE RADIUS OF A CIRCLE IS 12 UNITS. FIND THE PERIMETER OF A REGULAR INSCRIBED:

A TRIANGLE **B** HEPTAGON **C** DECAGON

5.2 FURTHER ON CONGRUENCY AND SIMILARITY

Congruency

TODAY, MODERN INDUSTRIES PRODUCE LARGE NUMBERS OF PRODUCTS; MANY OF THESE ARE THE SAME SIZE AND/OR SHAPE. TO DETERMINE THESE SHAPES AND SIZES, THE IDEA OF C IS VERY IMPORTANT.

TWO PLANE FIGURES ARE CONGRUENT IF THEY ARE EXACT COPIES OF EACH OTHER.

Group Work 5.1

1 LOOK CAREFULLY AT THE FIGURES GIVEN BELOW. LIST THE PAIRS THAT APPEAR TO BE CONGRUENT.

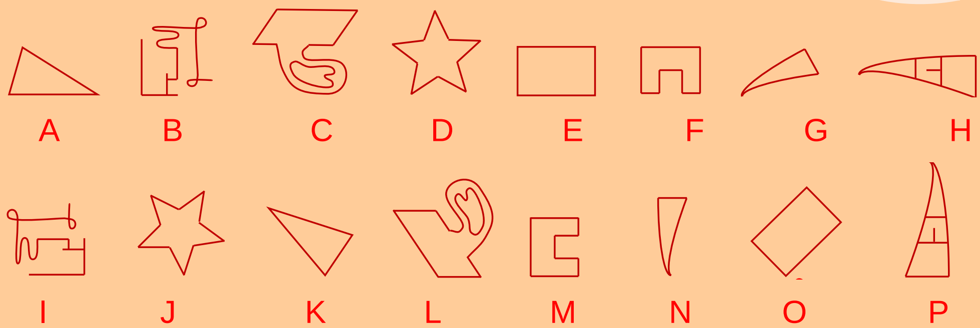


Figure 5.23

2 TEST WHETHER EACH PAIR IS, IN FACT, CONGRUENT BY TRACING ONE ON A THIN TRANSPARENT PAPER AND PLACING THE TRACING ON THE OTHER.

5.2.1 Congruency of Triangles

TRIANGLES THAT HAVE THE SAME SIZE AND SHAPE ARE CALLED **congruent triangles**. THAT IS, THE SIX PARTS OF THE TRIANGLES (THREE SIDES AND THREE ANGLES) ARE COR CONGRUENT. IF TWO TRIANGLES ABC AND DEF ARE CONGRUENT LIKE THOSE GIVEN BELOW, THE WE DENOTE THIS AS

$$\Delta ABC \cong \Delta DEF.$$

THE NOTATION "≅" MEANS "is congruent to".

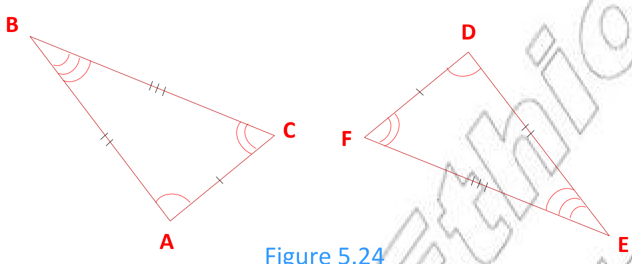


Figure 5.24

Congruent angles	Congruent sides
$\angle A \cong \angle D$; $\angle B \cong \angle E$; $\angle C \cong \angle F$	$\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$.

PARTS OF CONGRUENT TRIANGLES THAT "MATCH" ARE CALLED CORRESPONDING PART

THE TRIANGLES ABOVE, $\overline{AB}$ CORRESPONDS TO $\overline{DE}$ AND $\overline{AC}$ CORRESPONDS TO $\overline{DF}$.

TWO TRIANGLES ARE CONGRUENT WHEN ALL OF THE CORRESPONDING PARTS A HOWEVER, YOU DO NOT NEED TO KNOW ALL OF THE SIX CORRESPONDING PARTS TO THE TRIANGLES ARE CONGRUENT. EACH OF THE FOLLOWING THEOREMS STATE CORRESPONDING PARTS DETERMINE THE CONGRUENCE OF TWO TRIANGLES.

Congruent triangles

Two triangles are congruent if the following corresponding parts of the triangles are congruent.

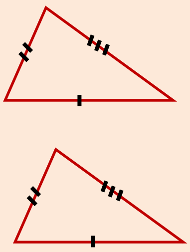
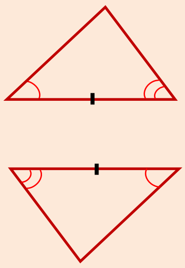
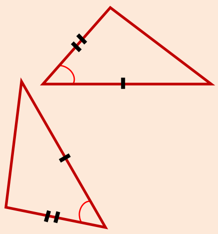
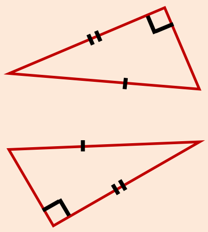
THREE SIDES (SSS)	TWO ANGLES AND INCLUDED SIDE (ASA)	TWO SIDES AND THE INCLUDED ANGLE (SAS)	A RIGHT ANGLE, HYPOTENUSE AND A SIDE (RHS)
			
A	B	C	D

Figure 5.25

EXAMPLE 1 DETERMINE WHETHER EACH PAIR OF TRIANGLES IS CONGRUENT. IF SO, WRITE A CONGRUENCE STATEMENT AND STATE WHY THE TRIANGLES ARE CONGRUENT.

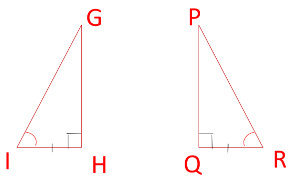


Figure 5.26

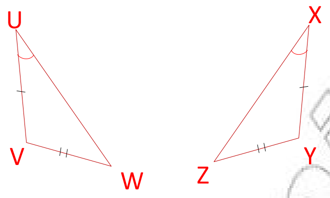


Figure 5.27

SOLUTION:

For the first two triangles: (FIGURE 5.26)

SINCE M(∠H) = 90^o AND M(∠Q) = 90^o, ∠H ≅ ∠Q

ALSO $\overline{IH} \cong \overline{RQ}$ (GIVEN)

∠I ≅ ∠R (GIVEN)

∴ Δ GHI ≅ Δ PQR (BY ASA)

For the last two triangles: (FIGURE 5.27)

$\overline{UV} \cong \overline{XY}$ (GIVEN)

∠VUW ≅ ∠YXZ (GIVEN)

$\overline{VW} \cong \overline{YZ}$ (GIVEN)

SO TWO SIDES AND AN ANGLE ARE CONGRUENT. BUT THE ANGLE IS NOT INCLUDED BETWEEN THE SIDES. SO WE CANNOT CONCLUDE THAT THE TRIANGLES ARE CONGRUENT.

EXAMPLE 2 IN FIGURE 5.28, PQRS IS A SQUARE. A AND B ARE POINTS ON $\overline{QR}$ AND $\overline{SR}$ SUCH THAT $\overline{QA} \cong \overline{SB}$.

PROVE THAT ∠PAQ ≅ ∠PBS

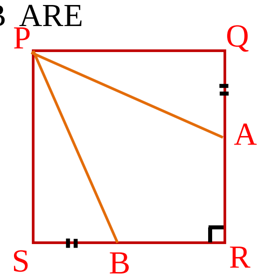


Figure 5.28

SOLUTION: $\overline{PQ} \cong \overline{PS}$ (*sides of a square*)

$\overline{QA} \cong \overline{SB}$ (*given*)

∠Q ≅ ∠S (*right angles*)

THEREFORE, ΔPQA ≅ ΔPSB (BY SAS).

THEREFORE, ∠PAQ ≅ ∠PBS (CORRESPONDING ANGLES OF CONGRUENT TRIANGLES).

EXAMPLE 3 GIVEN ΔABC ≅ ΔRST.

FIND THE VALUE OF y, IF M(∠A) = 40^O AND M(∠R) = $(2y + 10)^O$.

SOLUTION: SINCE ΔABC ≅ ΔRST, THE CORRESPONDING ANGLES ARE CONGRUENT.

SO, ∠A ≅ ∠R.

THEREFORE, M(∠A) = M(∠R).

I.E., $40^o = (2y + 10)^o$. SO, $y = 15^o$.

Exercise 5.3

1 CHECK WHETHER THE FOLLOWING FOUR TRIANGLES ARE CONGRUENT:

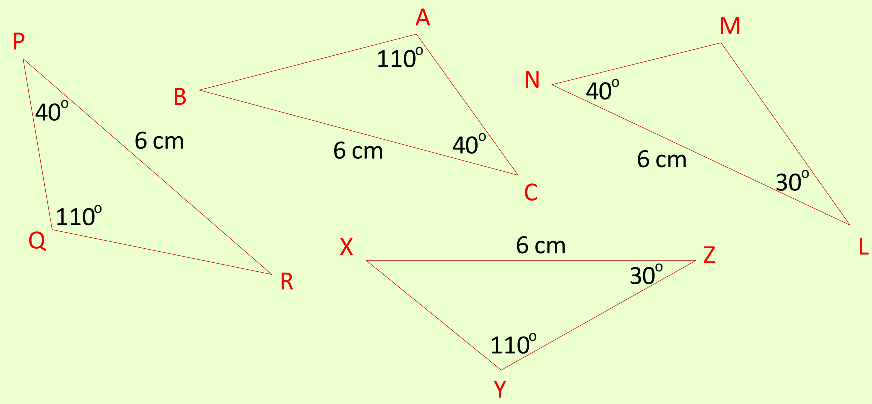


Figure 5.29

2 WHICH OF THE TRIANGLES ARE CONGRUENT TO THE BLUE-SHADED TRIANGLE? GIVE REASONS FOR YOUR ANSWER.

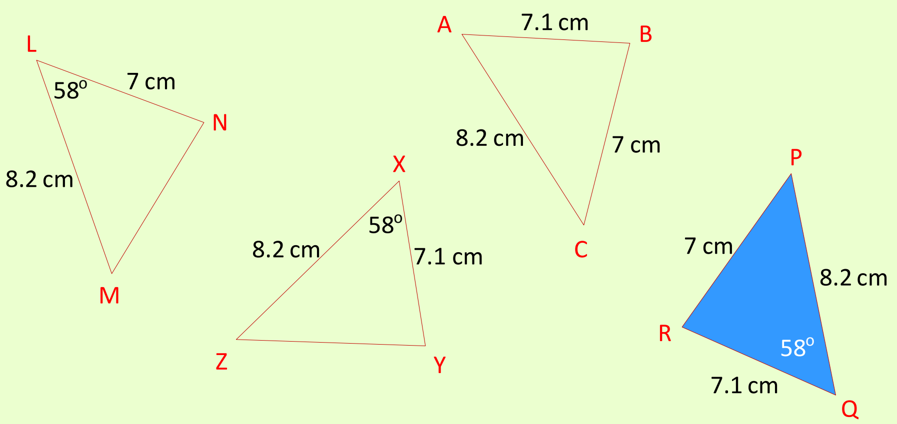


Figure 5.30

3 WHICH OF THE FOLLOWING PAIRS OF TRIANGLES ARE CONGRUENT? FOR THOSE THAT ARE CONGRUENT, STATE WHETHER THE REASON IS SSS, ASA, SAS OR RHS.

A

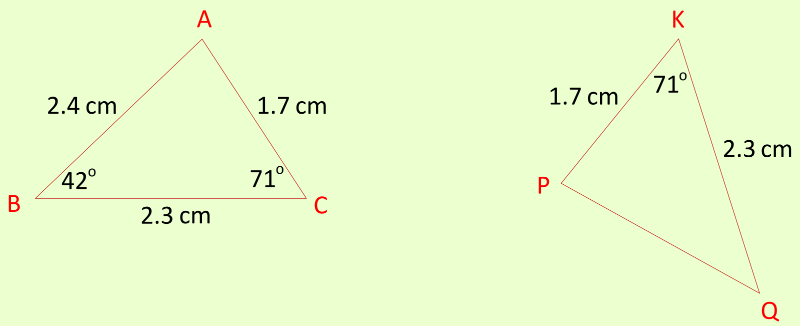


Figure 5.31

B

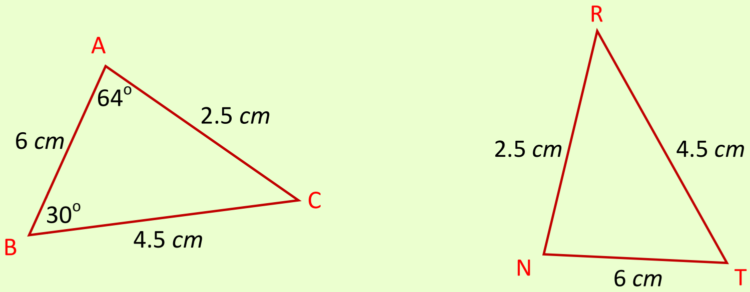


Figure 5.32

C

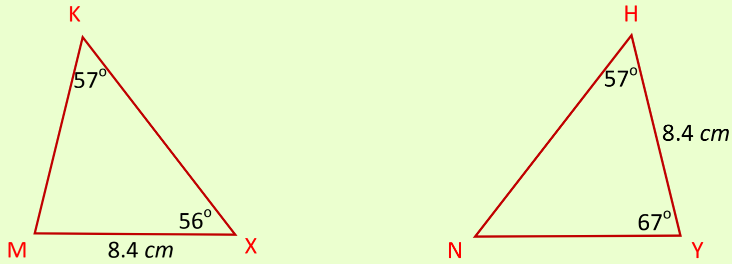


Figure 5.33

4 **A** ABC IS AN ISOSCELES TRIAN $\overline{AB} \cong \overline{AC}$, AND M IS THE MIDPO $\overline{BC}$. PROVE THAT $\angle ABC \cong \angle ACB$.

B IN FIGURE 5.3 BELOW, PROVE ΔBDF IS EQUILATERAL

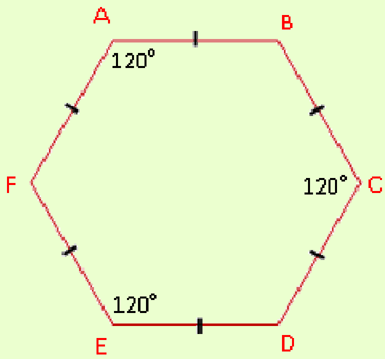


Figure 5.34

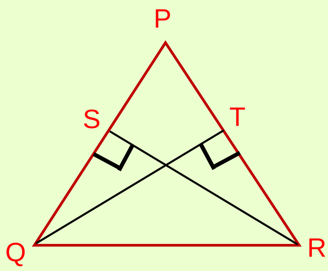


Figure 5.35

C IN FIGURE 5.3, PROV IF $\overline{RS} \cong \overline{QT}$ THEN, $\overline{PQ} \cong \overline{PR}$.

D ABC IS AN ISOSCELES TRIAN $\overline{AB} \cong \overline{AC}$. $\overline{AX}$ IS THE BISECT $\angle BAC$ MEETING BC AT X. PROVE THAT X IS THE MIDPOINT OI

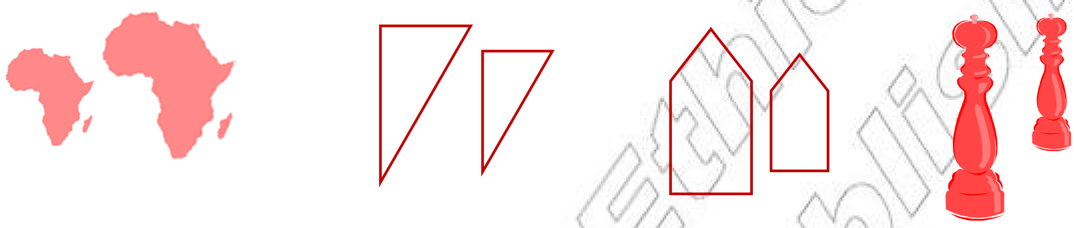
E ABCD IS A PARALLELOGRAM. P $\angle ABC \cong \angle ADC$.

Hint: FIRST JOIN AC AND USE ALTERNATE

5.2.2 Definition of Similar Figures

AFTER AN ARCHITECT FINISHES THE PLAN OF A BUILDING, IT IS USUAL TO PREPARE A BUILDING. IN DIFFERENT AREAS OF ENGINEERING, IT IS USUAL TO PRODUCE MODELS PRODUCTS BEFORE MOVING TO THE ACTUAL PRODUCTION. WHAT RELATIONSHIPS DO Y THE MODEL AND THE ACTUAL PRODUCT?

FIGURES THAT HAVE THE SAME SHAPE BUT THAT MIGHT HAVE DIFFERENT SIZES ARE C **similar.** EACH OF THE FOLLOWING PAIRS OF FIGURES ARE SIMILAR, WITH ONE SHAPE BEING AN OF THE OTHER.



A B C D

Figure 5.36

FROM YOUR GRADE 8 MATHEMATICS, RECALL THAT:

AN ENLARGEMENT IS A TRANSFORMATION OF A PLANE FIGURE IN WHICH EACH OF THE A, B, C IS MAPPED ONTO A', B', C' BY THE SAME SCALE FACTOR, k, FROM A FIXED POINT O. THE DISTANCES OF A', B', C' FROM THE POINT O ARE FOUND BY MULTIPLYING EACH OF THE D A, B, C FROM O BY THE SCALE FACTOR k.

$OA' = k \times OA$ $\qquad$ $OB' = k \times OB$ $\qquad$ $OC' = k \times OC$

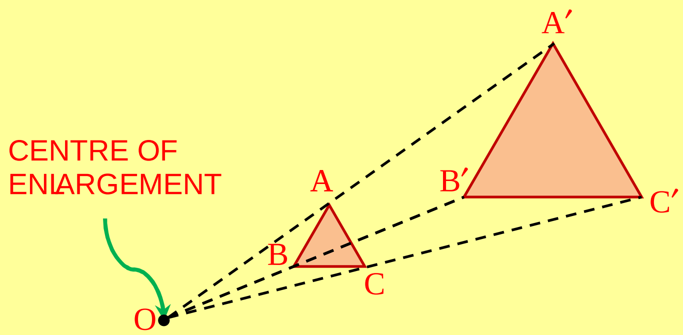


Figure 5.37

Group Work 5.2

1 ANSWER THE FOLLOWING QUESTIONS BASED ON FIGURE 5.38

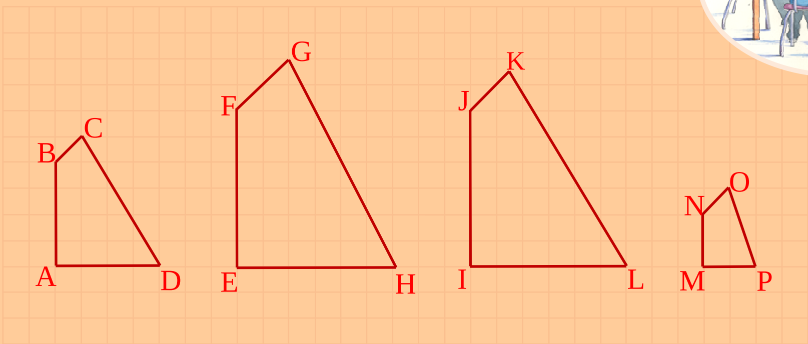


Figure 5.38

i IF A FIGURE IS ENLARGED, DO YOU ALWAYS GET A SIMILAR FIGURE?

ii WHICH OF THE FIGURES ABOVE ARE SIMILAR FIGURES? DISCUSS.

iii WHAT CAN YOU SAY ABOUT THE ANGLES B AND J?

iv WHICH OF THE ANGLES ARE EQUAL TO ANGLE C? WHICH OF THE ANGLES ARE EQUAL TO ANGLE D?

v WHAT OTHER EQUAL ANGLES CAN YOU FIND? DISCUSS.

vi WHAT CAN YOU SAY ABOUT THE ANGLES OF TWO SIMILAR FIGURES? DISCUSS.

2 FIGURES ABCDE AND FGHIJ ARE SIMILAR AND

$$\frac{BC}{GH} = \frac{8}{4} = 2.$$

FIND THE RATIO OF OTHER CORRESPONDING SIDES OF ABCDE AND FGHIJ.

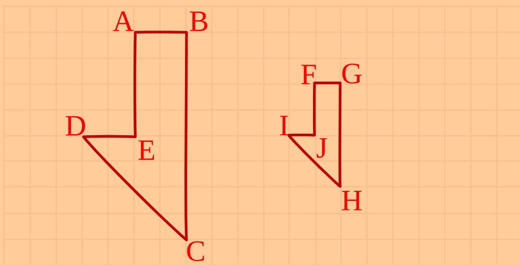


Figure 5.39

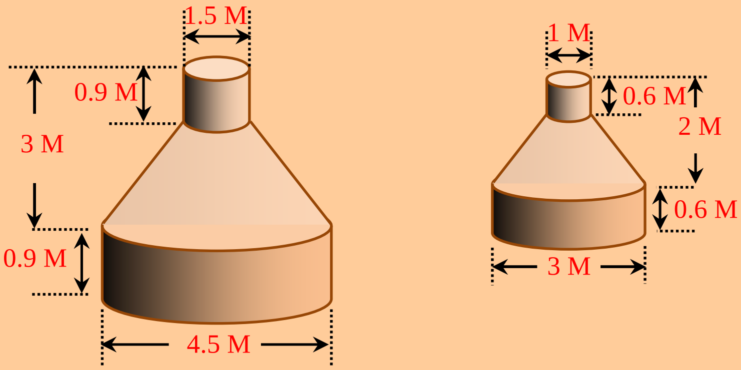


Figure 5.40

3 TWO SOLID FIGURES HAVE THE DIMENSIONS AS SHOWN IN FIGURE 5.40 ABOVE. ARE THE FIGURES SIMILAR? HOW CAN YOU MAKE SURE? DISCUSS.

4 **A** IS A RECTANGLE OF LENGTH 6 CM AND WIDTH 4 CM SIMILAR TO A RECTANGLE OF LENGTH 12 CM AND WIDTH 18 CM?

B IS A TRIANGLE, TWO OF WHOSE ANGLES ARE 72° AND 85° SIMILAR TO A TRIANGLE TWO OF WHOSE ANGLES ARE 23° AND 85°?

HOW COULD YOU HAVE ANSWERED PARTS A AND B OF THIS QUESTION WITHOUT DRAWING? DISCUSS.

FROM THE ABOVE GROUP WORK, WE MAY CONCLUDE THE FOLLOWING.

IN SIMILAR FIGURES:

- **I** ONE IS AN ENLARGEMENT OF THE OTHER.
- **II** ANGLES IN CORRESPONDING POSITIONS ARE CONGRUENT.
- **III** CORRESPONDING SIDES HAVE THE SAME RATIO.

IN THE CASE OF A POLYGON, THE ABOVE FACTS CAN BE STATED AS:

Similar polygons	TWO POLYGONS OF THE SAME NUMBER OF SIDES ARE SIMILAR, IF TH CORRESPONDING ANGLES ARE CONGRUENT AND THEIR CORRESPONDIN HAVE THE SAME RATIO.

EXAMPLE 1

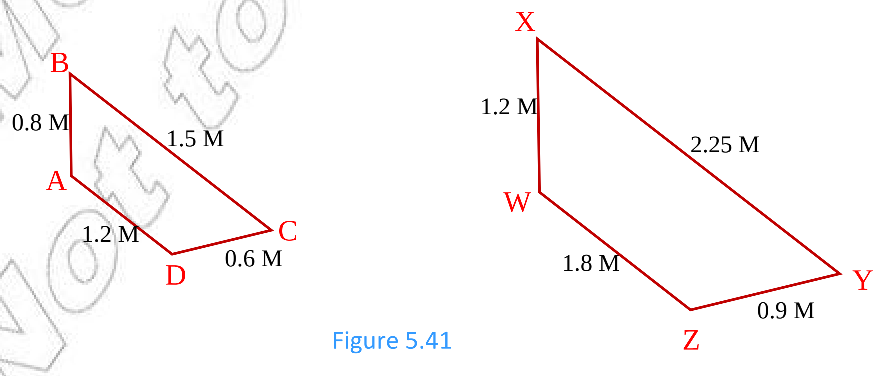


Figure 5.41

IF QUADRILATERAL ABCD IS SIMILAR TO QUADRILATERAL WXYZ, WE WRITE ABCD ~ WXYZ. (THE SYMBOL ~ MEANS "is similar to").

CORRESPONDING ANGLES OF SIMILAR POLYGONS ARE CONGRUENT. YOU CAN USE A PROTRACTOR TO MAKE SURE THE ANGLES HAVE THE SAME MEASURE.

$\angle A \cong \angle W \qquad \angle B \cong \angle X$

$\angle C \cong \angle Y \qquad \angle D \cong \angle Z$

USING CORRESPONDING ANGLES AS A GUIDE, YOU CAN EASILY IDENTIFY THE CORRESPONDING SIDES.

$\overline{AB} \leftrightarrow \overline{WX}, \overline{BC} \leftrightarrow \overline{XY}$

$\overline{CD} \leftrightarrow \overline{YZ}, \overline{DA} \leftrightarrow \overline{ZW}$

(THE SYMBOL $\leftrightarrow$ MEANS "CORRESPONDS TO").

A SPECIAL RELATIONSHIP ALSO EXISTS BETWEEN THE CORRESPONDING SIDES OF SIMILAR POLYGONS. COMPARE THE RATIOS OF LENGTHS OF THE CORRESPONDING SIDES:

$$\frac{AB}{WX}=\frac{0.8}{1.2}=\frac{2}{3},\ \frac{BC}{XY}=\frac{1.5}{2.25}=\frac{2}{3},\ \frac{CD}{YZ}=\frac{0.6}{0.9}=\frac{2}{3},\ \frac{DA}{ZW}=\frac{1.2}{1.8}=\frac{2}{3}.$$

YOU CAN SEE THE RATIOS OF THE LENGTHS OF THE CORRESPONDING SIDES ARE ALL EQUAL TO $\frac{2}{3}$.

EXAMPLE 2 REFERRING TO FIGURE 5.42, IF ABCDE ~ HIJKL, THEN FIND THE LENGTHS OF:

A $\overline{IJ}$ **B** $\overline{CD}$ **C** $\overline{HL}$

SOLUTION: SINCE ABCDE ~ HIJKL, WE HAVE,

$$\frac{AB}{HI}=\frac{BC}{IJ}=\frac{CD}{JK}=\frac{DE}{KL}=\frac{AE}{HL}$$

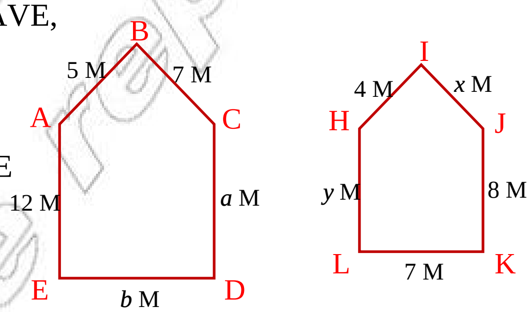


Figure 5.42

A TO FIND THE LENGTH OF $\overline{IJ}$, WE USE

$$\frac{AB}{HI}=\frac{BC}{IJ}$$

$\frac{5}{4}=\frac{7}{x}$ (AB = 5, HI = 4, BC = 7, IJ = x)

SO, $x=\frac{4\times 7}{5}=5.6$

THEREFORE, THE LENGTH OF $\overline{IJ}$ IS 5.6 M.

B IN THE SAME WAY,

$\frac{AB}{HI}=\frac{CD}{JK}$

$\frac{5}{4}=\frac{a}{8}$. SO, $a = 10$.

THEREFORE, CD = 10 M.

C $\frac{AB}{HI}=\frac{AE}{HL}$. I.E., $\frac{5}{4}=\frac{12}{y}$

SO, $y = 9.6$.

THEREFORE, HL = 9.6 M.

Exercise 5.4

1 **A** ALL OF THE FOLLOWING POLYGONS ARE REGULAR. IDENTIFY THE SIMILAR ONES.

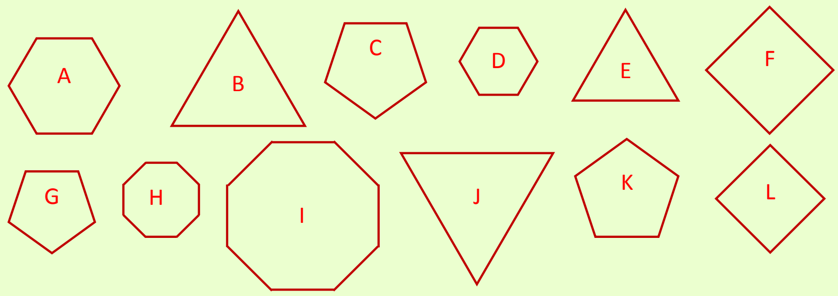


Figure 5.43

B EXPLAIN WHY REGULAR POLYGONS WITH THE SAME NUMBER OF SIDES ARE ALWAYS SIMILAR.

2 EXPLAIN WHY ALL CIRCLES ARE SIMILAR.

3 DECIDE WHETHER OR NOT EACH PAIR OF POLYGONS IS SIMILAR. EXPLAIN YOUR REASONING.

A

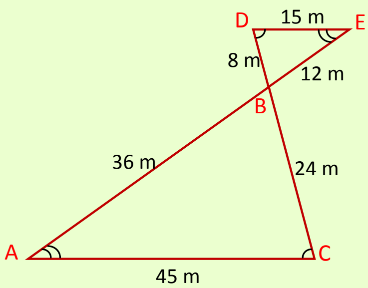


Figure 5.44

B

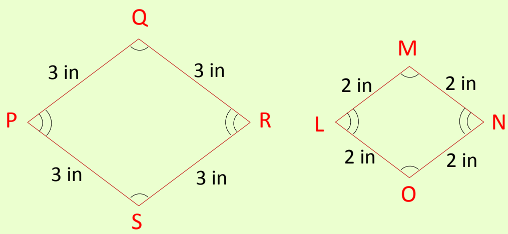


Figure 5.45

C

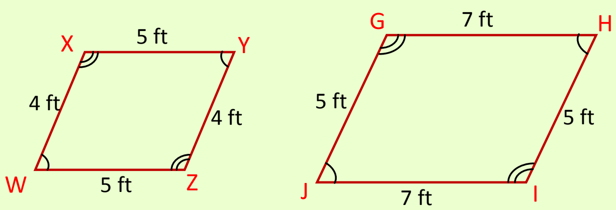


Figure 5.46

5.2.3 Theorems on Similarity of Triangles

YOU MAY START THIS SECTION BY RECALLING THE FOLLOWING FACTS ABOUT SIMILAR

Definition 5.5

Two triangles are said to be similar, if

1 their corresponding sides are proportional (have equal ratio), and

2 their corresponding angles are congruent.

That is, $\Delta ABC \sim \Delta DEF$ if and only if

$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ and

$\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$

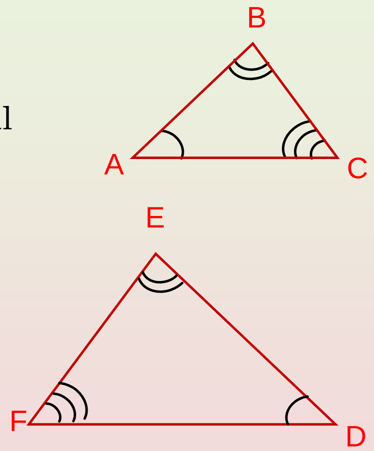


Figure 5.47

THE FOLLOWING THEOREMS ON SIMILARITY OF TRIANGLES WILL SERVE AS TESTS TO C
NOT TWO TRIANGLES ARE SIMILAR.

Theorem 5.4 **SSS similarity theorem**

If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.

Restatement:

GIVEN ΔABC AND ΔDEF. IF $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$,

THEN ΔABC ~ ΔDEF.

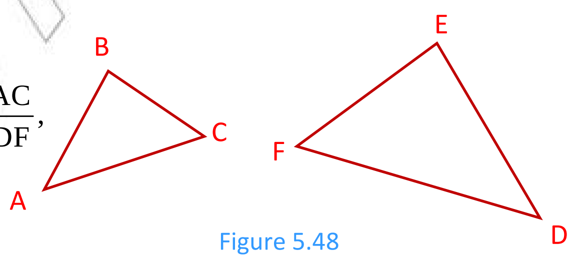


Figure 5.48

EXAMPLE 1 ARE THE TWO TRIANGLES IN FIGURE 5.49 SIMILAR?

SOLUTION: SINCE $\frac{PQ}{ST} = \frac{QR}{TU} = \frac{PR}{SU} = \frac{1}{2}$,

Δ PQR ~ ΔSTU (BY SSS SIMILARITY THEOREM).

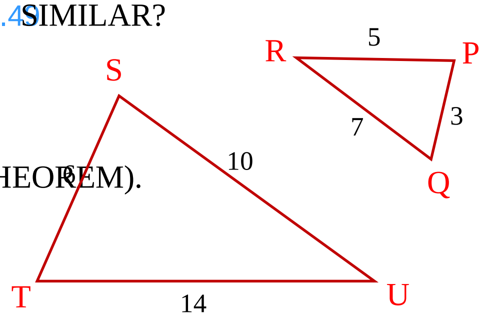


Figure 5.49

Group Work 5.3

1 INVESTIGATE IF THEOREM 5.3 WORKS FOR TWO POLYGON NUMBER OF SIDES IS GREATER THAN 3.

2 CONSIDER A SQUARE ABCD AND A RHOMBUS PQRS.

ARE THE RATIOS: $\frac{AB}{PQ}$, $\frac{BC}{QR}$ AND $\frac{CD}{RS}$ EQU

WE NOW STATE THE SECOND THEOREM ON SIMILARITY OF TRIANGLES, WHICH IS side-angle-side (SAS) SIMILARITY THEOREM.

Theorem 5.5 SAS similarity theorem

Two triangles are similar, if two pairs of corresponding sides of the two triangles are proportional and if the included angles between these sides are congruent.

Restatement:

GIVEN TWO TRIANGLES ΔABC AND ΔPQR, IF

$\frac{AB}{PQ} = \frac{AC}{PR}$ AND ∠ A ≅ ∠P, THEN,

ΔABC ~ ΔPQR.

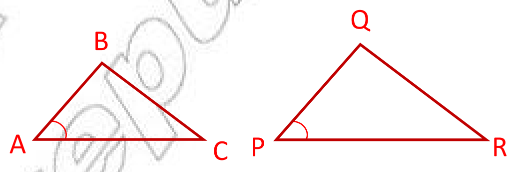


Figure 5.50

EXAMPLE 2 USE THE SAS SIMILARITY THEOREM TO CHECK WHETHER THE GIVEN TWO TRIANGLES ARE SIMILAR.

SOLUTION: SINCE $\frac{LN}{PR} = \frac{12}{6} = 2$, AND ALSO

$\frac{MN}{QR} = \frac{16}{8} = 2$, THE CORRESPONDING SIDES HAVE EQUAL RATIOS (I.E., THEY ARE PROPORTIONAL).

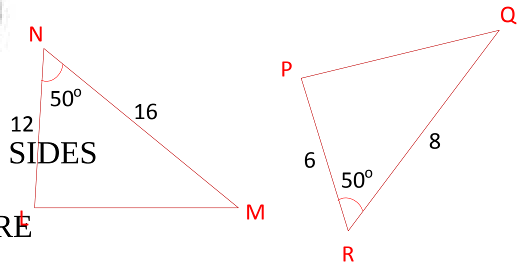


Figure 5.51

ALSO, SINCE M(∠N) = M(∠R), IT FOLLOWS THAT THE INCLUDED ANGLES OF THE PROPORTIONAL SIDES ARE CONGRUENT.

THEREFORE, ΔLMN ~ ΔPQR BY THE SAS SIMILARITY THEOREM.

FINALLY, WE STATE THE THIRD THEOREM ON SIMILARITY OF TRIANGLES, WHICH Angle-Angle (AA) SIMILARITY THEOREM.

Theorem 5.6 AA similarity theorem

If two angles of one triangle are congruent to two corresponding angles of another triangle, then the two triangles are similar.

Restatement:

GIVEN TWO TRIANGLES, NAMELY ΔABC AND ΔDEF. IF $\angle A \cong \angle D$ AND $\angle C \cong \angle F$, THEN $\Delta ABC \sim \Delta DEF$.

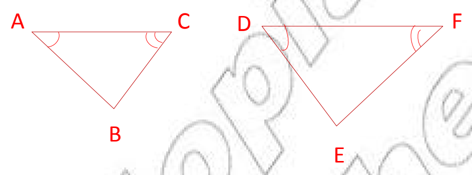


Figure 5.52

EXAMPLE 3 IN FIGURE 5.53, DETERMINE WHETHER THE TWO GIVEN TRIANGLES ARE SIMILAR.

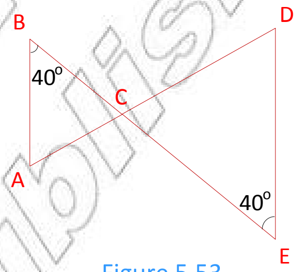


Figure 5.53

SOLUTION: IN ΔABC AND ΔDEC, $m(\angle B) = m(\angle E) = 40^o$.

SO, **i** $\angle B \cong \angle E$.

ii $\angle ACB \cong \angle DCE$ (SINCE THEY ARE VERTICALLY OPPOSITE ANGLES).

THEREFORE, $\Delta ABC \sim \Delta DEC$ BY THE AA SIMILARITY THEOREM.

Exercise 5.5

1 STATE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE.

a IF TWO TRIANGLES ARE SIMILAR, THEN THEY ARE CONGRUENT.

b IF TWO TRIANGLES ARE CONGRUENT, THEN THEY ARE SIMILAR.

c ALL EQUILATERAL TRIANGLES ARE CONGRUENT.

d ALL EQUILATERAL TRIANGLES ARE SIMILAR.

2 WHICH OF THE FOLLOWING PAIRS OF TRIANGLES ARE SIMILAR? IF THEY ARE SIMILAR, EXPLAIN WHY.

a

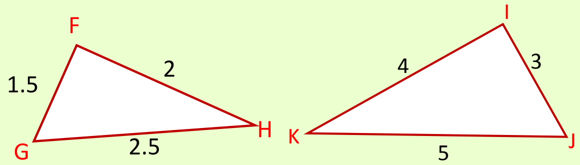


Figure 5.54

b

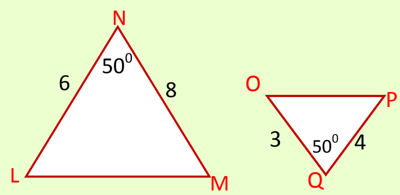


Figure 5.55

3 THE PAIRS OF TRI GIVEN BELOW ARE SIMILAR. FIND THE MEASURES BLANK SIDES.

A

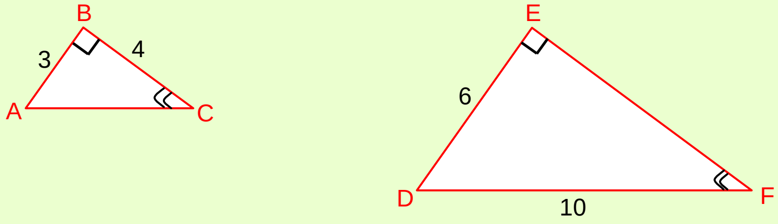


Figure 5.56

B

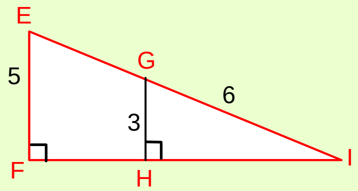


Figure 5.57

4 IN FIGURE 5.58, PROVE T:

A Δ ADC ~ Δ BEC

B Δ AFE ~ Δ BFD

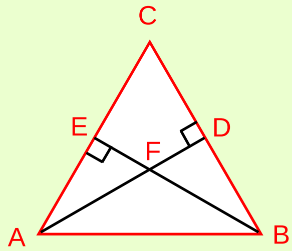


Figure 5.58

5 IN FIGURE 5.59, QUADRILA DEFG IS A SQUARE AND, ∠C IS A RIGHT ANGLE. PR:

A $\dfrac{AD}{EF} = \dfrac{DG}{EB}$

B $\dfrac{AD}{CG} = \dfrac{DG}{CF}$

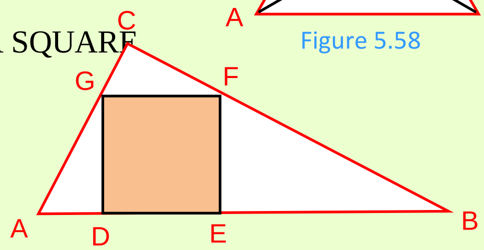


Figure 5.59

5.2.4 Theorems on Similar Plane Figures

Ratio of perimeters and ratio of areas of similar plane figures

ACTIVITY 5.7

CONSIDER FIGURE 5.60 GIVEN BELOW ANSWER THE FOLLOWING (

A SHOW THAT THE TWO RECTANGL

B WHAT IS THE RATIO OF THE CORRESI

C FIND THE PERIME AND THE AREA OF EACH RECTANG

D WHAT IS THE RATIO OF PERIMETERS?

E WHAT IS THE RATIO OF THE

F WHAT IS THE RELATIONSHIP BETWEEN THE RATIO OF THE CORRESPONDING SIDES AND THE RATIO OF THE PERIMETERS?

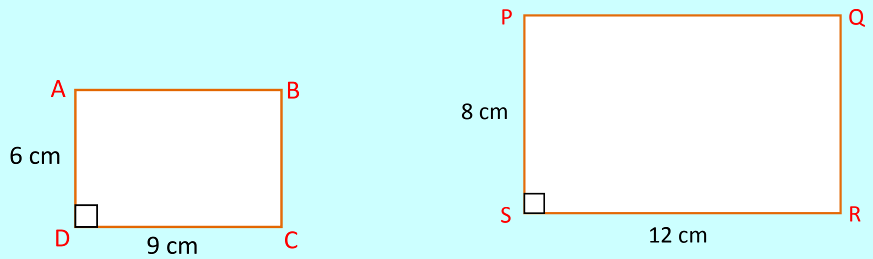


Figure 5.60

G WHAT IS THE RELATIONSHIP BETWEEN THE RATIO OF THE CORRESPONDING SIDES AND RATIO OF THE AREAS?

THE RESULTS OF THE ACTIVITY LEAD YOU TO THE FOLLOWING THEOREM.

Theorem 5.7

If the ratio of the lengths of the corresponding sides of two similar triangles is k, then

I the ratio of their perimeters is k

II the ratio of their areas is k^2.

Proof:-

I GIVEN $\Delta ABC \sim \Delta PQR$.

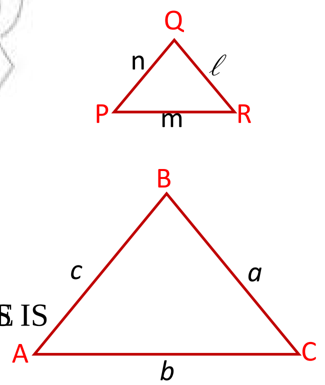


Figure 5.61

THEN, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

I.E., $\frac{c}{n} = \frac{a}{l} = \frac{b}{m}$.

SINCE THE COMMON VALUE OF THESE RATIOS IS k, WE HAVE

$$\frac{c}{n} = \frac{a}{l} = \frac{b}{m} = k.$$

SO, $c = kn$, $a = kl$, $b = km$.

NOW THE PERIMETER OF ΔABC = AB + BC + CA = $c + a + b = kn + kl + km$.

FROM THIS, WE OBTAIN $c + a + b = kn + kl + km = k(n + l + m)$

THEREFORE, $\frac{c+a+b}{n+l+m} = k$.

THAT IS, $\frac{\text{PERIMETER OF } \Delta ABC}{\text{PERIMETER OF } \Delta PQR} = k$.

THIS SHOWS THAT THE RATIO OF THEIR PERIMETERS = THE RATIO OF THEIR CORRESPON

II TO PROVE THAT THE RATIO OF THEIR AREAS IS THE SQUARE OF THE RATIO OF ANY TWO CORRESPONDING SIDES:

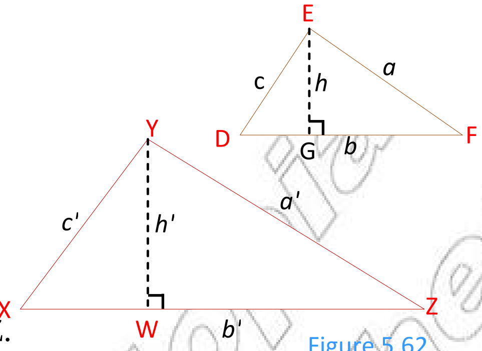


Figure 5.62

LET ΔDEF ~ ΔXYZ. THEN,

$$\frac{DE}{XY} = \frac{EF}{YZ} = \frac{DF}{XZ} = k.$$

THAT IS, $\frac{c}{c'} = \frac{a}{a'} = \frac{b}{b'} = k$.

LET $\overline{EG}$ BE THE ALTITUDE FROM E TO $\overline{DF}$ AND $\overline{YW}$ BE THE ALTITUDE FROM Y TO XZ.

SINCE ΔDEG AND Δ XYW ARE RIGHT TRIANGLES AND ∠D ≅ ∠X, WE HAVE

ΔDEG ~ ΔXYW (AA SIMILARITY)

THEREFORE, $\frac{h}{h'} = \frac{c}{c'} = k$.

NOW, AREA OF ΔDEF = $\frac{1}{2}bh$, AND AREA OF ΔXYZ = $\frac{1}{2}b'h'$.

THEREFORE, $\frac{\text{AREA OF } \Delta DEF}{\text{AREA OF } \Delta XYZ} = \frac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \frac{b}{b'} \times \frac{h}{h'} = k \times k = k^2$.

SO, THE RATIO OF THEIR AREAS IS THE SQUARE OF THE RATIO OF THEIR CORRESPONDIN

NOW WE STATE THE SAME FACT FOR ANY TWO POLYGONS.

Theorem 5.8

If the ratio of the lengths of any two corresponding sides of two similar polygons is k, then

I the ratio of their perimeters is k.

II the ratio of their areas is k^2.

Exercise 5.6

1 LET ABCD AND EFGH BE TWO QUADRILATERALS SUCH THAT ABCD ~ EFGH.
IF AB = 15 CM, EF = 18 CM AND THE PERIMETER OF ABCD IS 40 CM, FIND THE PERIMETER OF EFGH.

2 TWO TRIANGLES ARE SIMILAR. A SIDE OF ONE IS 2 UNITS LONG. THE CORRESPONDING SIDE OF THE OTHER IS 5 UNITS LONG. WHAT IS THE RATIO OF:

A THEIR PERIMETERS? **B** THEIR AREAS?

3 TWO TRIANGLES ARE SIMILAR. THE SIDES OF ONE ARE 3 TIMES AS LONG AS THE SIDES OF THE OTHER. WHAT IS THE RATIO OF THE AREAS OF THE SMALLER TO THE LARGER?

4 THE AREAS OF TWO SIMILAR TRIANGLES ARE 144 UNIT^2 AND 81 UNIT^2.

- **A** WHAT IS THE RATIO OF THEIR PERIMETERS?
- **B** IF A SIDE OF THE FIRST IS 6 UNITS LONG, WHAT IS THE LENGTH OF THE CORRESPONDING SIDE OF THE SECOND?

5 THE SIDES OF A POLYGON HAVE LENGTHS 5, 7, 8, 11 AND 19 UNITS. THE PERIMETER OF A SIMILAR POLYGON IS 75 UNITS. FIND THE LENGTHS OF THE SIDES OF THE LARGER P

5.2.5 Construction of Similar Figures

Enlargement

Group Work 5.4

Work with a partner

1 DRAW A TRIANGLE ABC ON SQUARED PAPER AS SHOWN.

2 TAKE A POINT O AND DRAW RAYS $\overrightarrow{OA}$, $\overrightarrow{OB}$ AND $\overrightarrow{OC}$; ON THESE RAYS MARK POINTS A′, B′ AND C′ SUCH THAT OA′ = 2OA; OB′ = 2 OB; OC′ = 2 OC

3 WHAT CAN YOU SAY ABOUT ΔABC AND ΔA′B′C′?

4 IS $\frac{OA'}{OA} = \frac{A'B'}{AB}$?

5 WHAT PROPERTIES HAVE NOT CHANGED?

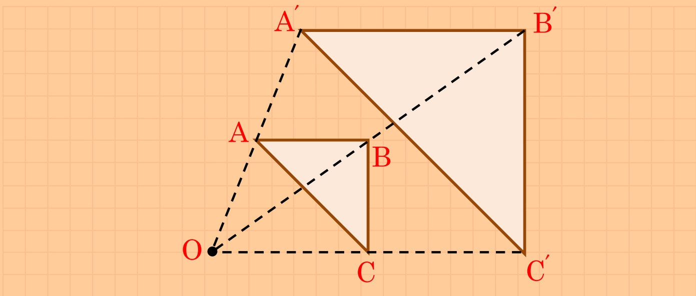


Figure 5.63

FIGURE 5.63 SHOWS TRIANGLE ABC AND ITS IMAGE TRIANGLE A′B′C′ UNDER THE TRANSFORMATION ENLARGEMENT. IN THE EQUATION OA′ = 2OA, THE FACTOR 2 IS CALLED THE scale factor AND THE POINT O IS CALLED the centre of enlargement.

IN GENERAL,

AN ENLARGEMENT WITH CENTRE O AND SCALE FACTOR k (WHERE k IS A REAL NUMBER) IS THE TRANSFORMATION THAT MAPS EACH POINT P TO A POINT P′ SUCH THAT

- **I** P′ IS ON THE RAY $\overrightarrow{OP}$ AND
- **II** O P′ = k OP

IF AN OBJECT IS ENLARGED, THE RESULT IS AN IMAGE THAT IS MATHEMATICALLY SIMI BUT OF DIFFERENT SIZE. THE IMAGE CAN BE EITHER LARGER, IF $k > 1$ OR SMALLER, IF $k < 1$.

EXAMPLE 1 IN FIGURE 5.64 BELOW, ΔABC IS ENLARGED TO FORM ΔA′B′C′. FIND THE CENTRE OF ENLARGEMENT.

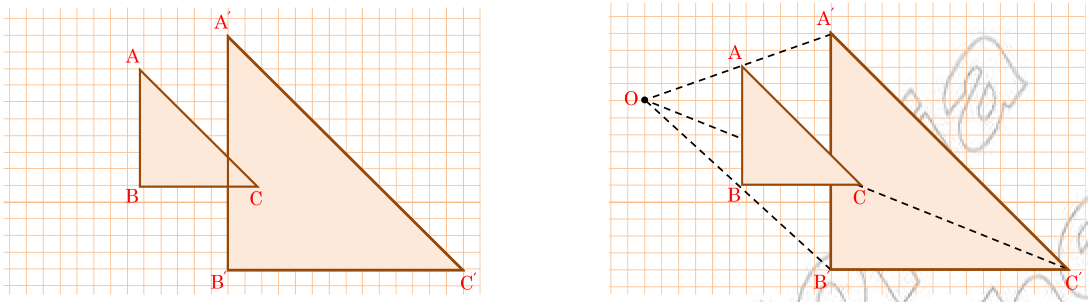


Figure 5.64

SOLUTION: THE CENTRE OF ENLARGEMENT IS FOUND BY JOINING CORRESPONDING POINTS ON THE OBJECT AND IMAGE WITH STRAIGHT LINES. THESE LINES ARE THEN EXTENDED TO MEET. THE POINT AT WHICH THEY MEET IS THE centre of enlargement O (*See* FIGURE 5.64B ABOVE).

EXAMPLE 2 IN FIGURE 5.65 BELOW, THE RECTANGLE ABCD UNDERGOES A TRANSFORMATION TO FORM RECTANGLE A′B′C′D′.

I FIND THE CENTRE OF ENLARGEMENT.

II CALCULATE THE SCALE FACTOR OF ENLARGEMENT.

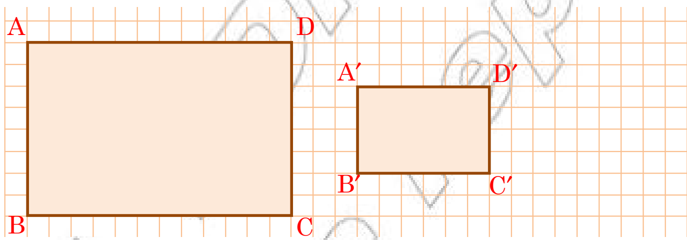


Figure 5.65

SOLUTION:

I BY JOINING CORRESPONDING POINTS ON BOTH THE OBJECT AND THE IMAGE, THE CENTRE OF ENLARGEMENT IS FOUND AT O, AS SHOWN IN FIGURE 5.66 BELOW.

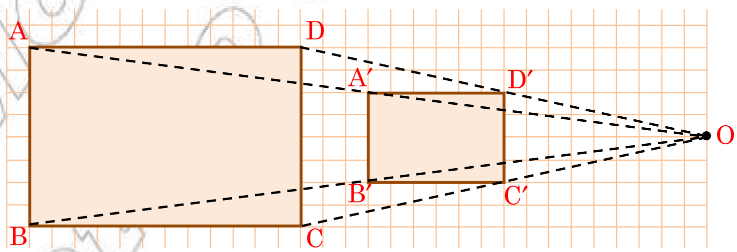


Figure 5.66

II THE SCALE FACTOR OF ENLARGEMENT $= \frac{A'B'}{AB} = \frac{4}{8} = \frac{1}{2}$

IF THE SCALE FACTOR OF ENLARGEMENT IS GREATER THAN 1, THEN THE IMAGE IS L OBJECT. IF THE SCALE FACTOR LIES BETWEEN 0 AND 1 THEN THE RESULTING IMAGE IS THE OBJECT. IN THESE LATTER CASES, ALTHOUGH THE IMAGE IS SMALLER THAN T TRANSFORMATION IS STILL KNOWN AS AN ENLARGEMENT.

Exercise 5.7

1 COPY THE FOLLOWING FIGURES AND FIND:

I THE CENTRE OF ENLARGEMENT. **II** THE SCALE FACTOR OF THE ENLARGEMENT.

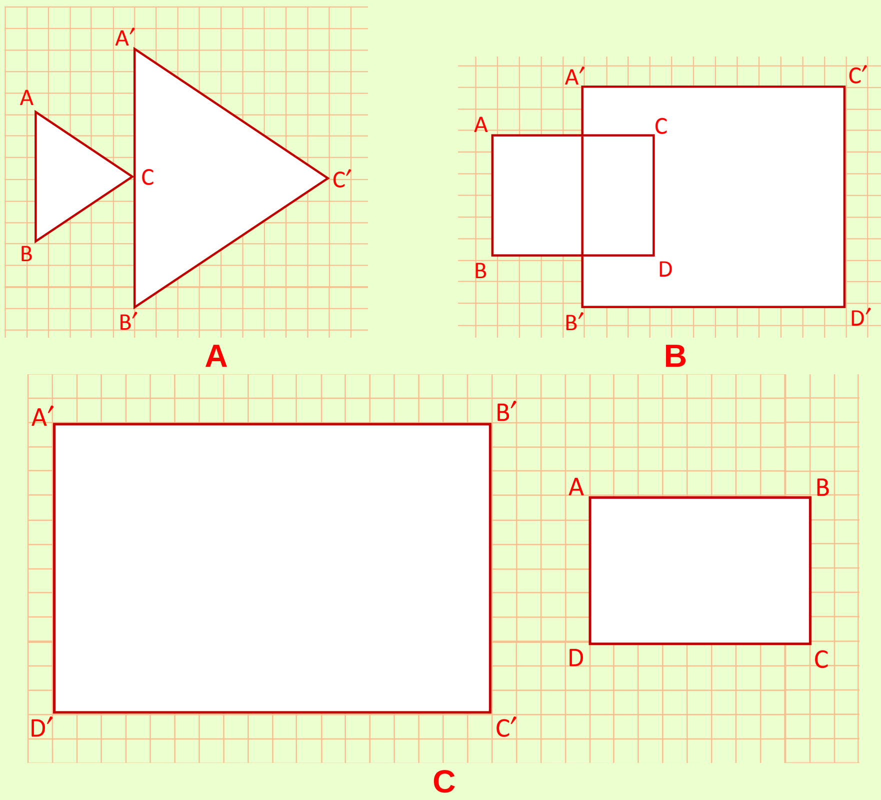


Figure 5.67

2 COPY AND ENLARGE EACH OF THE FOLLOWING FIGURES BY A SCALE FACTOR OF:

I 3 **II** $\frac{1}{2}$

(*O is the centre of enlargement*).

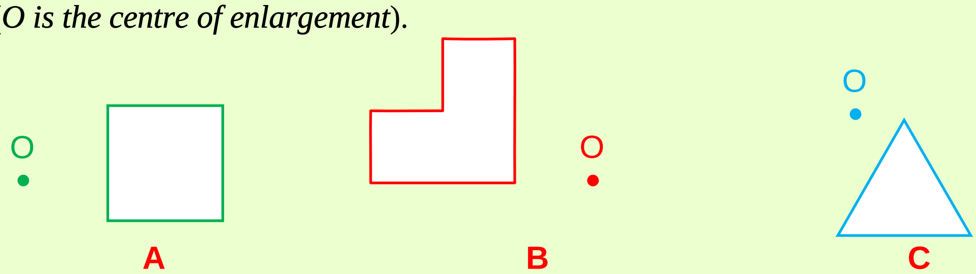


Figure 5.68

5.2.6 Real-Life Problems Using Congruency and Similarity

THE PROPERTIES OF CONGRUENCY AND SIMILARITY CAN BE APPLIED TO SOLVE SOME REAL-LIFE PROBLEMS AND ALSO TO PROVE CERTAIN GEOMETRIC PROPERTIES. FOR EX... FOLLOWING EXAMPLES.

EXAMPLE 1 SHOW THAT THE DIAGONALS OF A REC... CONGRUENT (FIGURE 5.69).

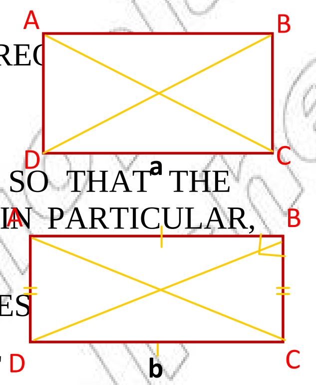


Figure 5.69

SOLUTION: SUPPOSE ABCD IS A RECTANGLE (FIGURE 5.69B).

THEN, ABCD IS A PARALLELOGRAM (WHY?) SO THAT THE OPPOSITE SIDES OF ABCD ARE CONGRUENT. IN PARTICULAR, $\overline{AB} \cong \overline{DC}$. CONSIDER ΔABC AND ΔDCB.

CLEARLY, $\angle ABC \cong \angle DCB$ (BOTH ARE RIGHT ANGLES).

HENCE, $\Delta ABC \cong \Delta DCB$ BY THE SAS CONGRUENCE PROPERTY. CONSEQUENTLY, $\overline{AC} \cong \overline{DB}$, AS DESIRED.

CARPENTERS USE THE RESULT OF EXAMPLE 1 WHEN FRAMING RECTANGULAR SHAPES. THAT IS, T... DETERMINE WHETHER A QUADRILATERAL IS A RECTANGLE, A CARPENTER CAN MEASU... TO SEE IF THEY ARE CONGRUENT (IF SO, THE SHAPE IS A PARALLELOGRAM). THEN THE... MEASURE THE DIAGONALS TO SEE IF THEY ARE CONGRUENT (IF SO, THE SHAPE IS A REC...

EXAMPLE 2 WHEN ALI PLANTED A TREE 5 M AWAY FROM POINT A, IT BLOCKED THE VIEW OF A BUILDING 50 M AWAY. IF THE BUILDING WAS 20 M TALL, HOW TA... THE TREE?

SOLUTION: LABEL THE FIGURE AS SHOWN. LET x BE THE HEIGHT OF THE TREE.

WE HAVE $\frac{BE}{CD} = \frac{AE}{AD}$

$$\frac{x}{20} = \frac{5}{50}$$

$$x = 20 \times \frac{5}{50} = 2 \text{ M}$$

∴ THE HEIGHT OF THE TREE WAS 2 M.

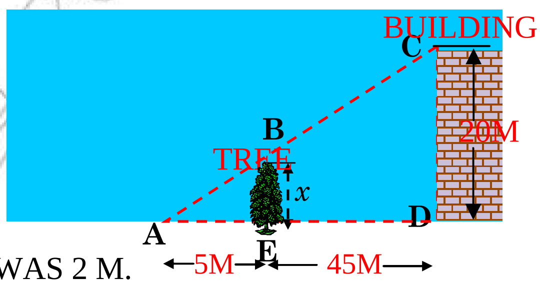


Figure 5.70

Exercise 5.8

1. AWEKE TOOK 1 HOUR TO CUT THE GRASS IN A SQUARE FIELD OF SIDE 30 M. HOW LONG WILL IT TAKE HIM TO CUT THE GRASS IN A SQUARE FIELD OF SIDE 120 M?
2. A LINE FROM THE TOP OF A CLIFF TO THE GROUND PASSES JUST OVER THE TOP OF A POLE 20 M HIGH. THE LINE MEETS THE GROUND AT A POINT 15 M FROM THE BASE OF THE PO... 120 M AWAY FROM THIS POINT TO THE BASE OF THE CLIFF, HOW HIGH IS THE CLIFF?
3. A TREE CASTS A SHADOW OF 30 M. AT THE SAME TIME, A 1 M POLE CASTS A SHADOW OF 12 M. FIND THE HEIGHT OF THE TREE.

5.3 FURTHER ON TRIGONOMETRY

5.3.1 Radian Measure of an Angle

AN ANGLE IS THE UNION OF TWO RAYS WITH A COMMON END POINT.

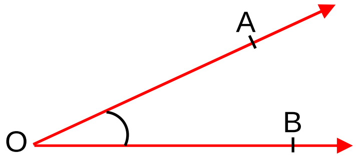


Figure 5.71

IN GENERAL, WE ASSOCIATE EACH ANGLE WITH A REAL NUMBER CALLED THE measure of the angle. THE TWO MEASURES THAT ARE MOST FREQUENTLY USED ARE DEGREE AND RADIAN.

I Measuring angles in degrees

WE KNOW THAT A RIGHT ANGLE CONTAINS 90°, AND THAT A COMPLETE ROTATION CAN BE THOUGHT OF AS AN ANGLE OF 360°. IN VIEW OF THIS LATTER FACT, WE CAN DEFINE A DEGREE AS FOLLOWS

Definition 5.6

A degree, denoted by (°), is defined as the measure of the central angle subtended by an arc of a circle equal in length to $\frac{1}{360}$ of the circumference of the circle.

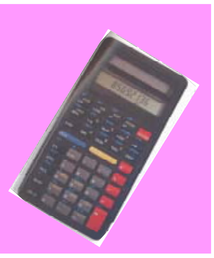
- ✓ A minute WHICH IS DENOTED BY (′), IS $\frac{1}{60}$ OF A DEGREE.
- ✓ A second WHICH IS DENOTED BY (″), IS $\frac{1}{60}$ OF A MINUTE.

SO, WE HAVE THE FOLLOWING RELATIONSHIP.

$1' = \left(\frac{1}{60}\right)^{o}$, $1'' = \left(\frac{1}{60}\right)'$ THAT IS $1'' = \left(\frac{1}{3600}\right)^{o}$ OR $1^{o} = 60'$ AND $1' = 60''$

Calculator Tip

Use your calculator to convert 20° 41'16'', which is read as 20 degrees, 41 minutes and 16 seconds, into degrees, (as a decimal).



II Measuring angles in radians

ANOTHER UNIT USED TO MEASURE ANGLES IS THE RADIAN. TO UNDERSTAND WHAT IS MEANT BY A RADIAN, WE AGAIN START WITH A CIRCLE. WE MEASURE A LENGTH OF r EQUAL TO THE RADIUS OF THE CIRCLE ALONG THE CIRCUMFERENCE OF THE CIRCLE, SO THAT ARC $\widehat{AB}$ IS EQUAL TO THE RADIUS r. $\angle AOB$ IS THEN AN ANGLE OF 1 RADIAN. WE DEFINE THIS AS FOLLOWS.

Definition 5.7

A radian (rad) is defined as the measure of the central angle subtended by an arc of a circle equal in length to the radius of the circle.

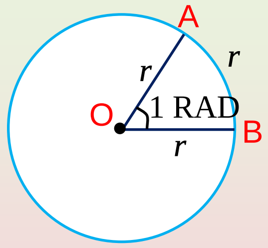


Figure 5.72

YOU KNOW THAT THE CIRCUMFERENCE OF A CIRCLE IS EQUAL TO $2 r$. SINCE AN ARC OF LENGTH r ALONG THE CIRCLE GIVES 1 RAD, A COMPLETE ROTATION OF LENGTH $2 r$ GENERATES AN ANGLE OF 2 RADIANS. ON THE OTHER HAND, WE KNOW THAT A COMPLETE REVOLUTION REPRESENTS 360^o. THIS GIVES US THE FOLLOWING RELATIONSHIP:

1 revolution = 360° = 2   radians

I.E., 180^o =   RADIANS, FROM WHICH WE OBTAIN,

1 RADIAN $= \left(\frac{180}{\ }\right)^o \approx 57.3^o$. $1^O = \frac{\ }{180}$ RADIAN ≈ 0.0175 RADIAN.

THEREFORE, WE HAVE THE FOLLOWING CONVERSION RULES FOR DEGREES AND RADIANS.

TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^O}{\ }$

TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\ }{180^O}$

EXAMPLE 1

I CONVERT EACH OF THE FOLLOWING TO RADIANS:

A 30^o **B** 90^o

II CONVERT EACH OF THE FOLLOWING TO DEGREES:

A $\frac{\ }{4}$ RAD **B** $\frac{\ }{3}$ RAD

SOLUTION:

I **A** $30^o = 30^O \times \frac{\ }{180^O} = \frac{\ }{6}$ RAD. **B** $90^o = 90^O \times \frac{\ }{180^O} = \frac{\ }{2}$ RAD.

II **A** $\frac{\ }{4}$ RAD $= \frac{\ }{4} \times \frac{180^O}{\ } = 45^O$. **B** $\frac{\ }{3}$ RAD $= \frac{\ }{3} \times \frac{180^O}{\ } = 60^O$.

5.3.2 Trigonometrical Ratios to Solve Right-angled Triangles

ACTIVITY 5.8

1 WHAT IS THE MEANA TRIGONOMETRIC RATIO?

2 GIVEN RIGHT-ANGTRIANG, ΔABC ANDΔA'B'C',

IF M(∠A) = M(∠A'), WHAT CAN YOU SAY ABOUT THE T

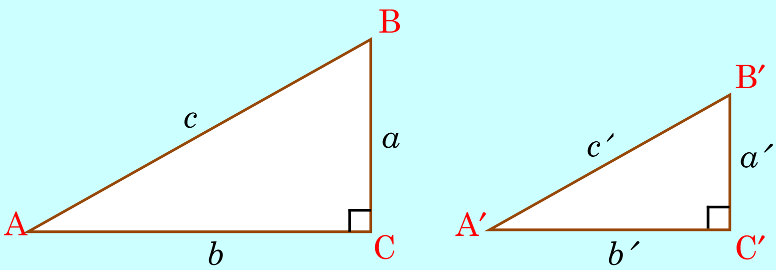


Figure 5.73

THE ANSWERSTHESE QUESTSHOULD HAVE LEAD YOUTHE RELATIONSHIPS T BETWEEN AN ANGLE AND THE SIDES OF A RIGHT-ANGLED TRIANGLE, WHICH HELP YOU TO SOLVE PROBLEMS THAT INVO-ANGLED TRIANGLES.

CONSIDER THE TWO TRIAFIGURE 5.73ABOVE.

GIVEN M(∠A) = M (∠A')

I ∠A ≅ ∠A'

II ∠C ≅ ∠C'

THEREFORE, ΔABC ~ ΔA'B'C' (BY AA SIMILARITY)

THIS MEANS $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$

FROM THIS WE GET,

1 $\frac{BC}{AB} = \frac{B'C'}{A'B'}$ 2 $\frac{AC}{AB} = \frac{A'C'}{A'B'}$ 3 $\frac{BC}{AC} = \frac{B'C'}{A'C'}$

OR $\frac{a}{c} = \frac{a'}{c'}$, $\frac{b}{c} = \frac{b'}{c'}$ AND $\frac{a}{b} = \frac{a'}{b'}$

THE FRACTIONS OR RATIOS IN EACH OF THESE PROtrigonometric ratios.

Sine:- THE FRACTIONS IN PRC1 ABOVE ARE FORMED BY DIVIDING THE OF∠A (OR∠A') BY THE HYPOTENUSE OF EAC THIS RATIO IS CALI SINE OF∠A IT IS ABBREVITO SIN A.

Cosine:- THE FRACTIONS IN PRO 2 ARE FORMED BY DIVIDING THE adjacent side TO ∠A (OR ∠A') BY TH hypotenuse OF EACH TRIANGLE. THIS RATIO IS CALL cosine OF ∠A. IT IS ABBREVIATED TO COS A.

Tangent:- THE FRACTION ROPORTION 3 ARE FORMED BY DIVIDING THE opposite side OF ∠A (OR ∠A') BY TH adjacent side. THIS RATIO IS CALLED THE ∠ A. IT IS ABBREVI TO TAN A.

THE FOLLOWING ABBREVIATIONS ARE CO

ADJ = ADJACENT S　　HYP = HYPOTENUSE　　**OPP** = OPPOSI SIDE.

THE ABOVE DISCUSSION CAN BE SUMMARIZED AND EXP

$\text{SIN A} = \frac{\text{OPP}}{\text{HYP}} = \frac{\text{BC}}{\text{AB}};$　　$\text{COS A} = \frac{\text{ADJ}}{\text{HYP}} = \frac{\text{AC}}{\text{AB}}$

$\text{TAN A} = \frac{\text{OPP}}{\text{ADJ}} = \frac{\text{BC}}{\text{AC}}$

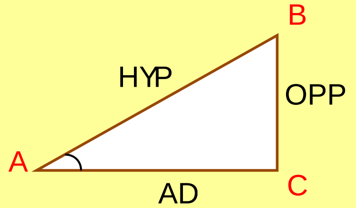


Figure 5.74

EXAMPLE 1 IN THE FOLLOWING RIGH, FIND THE VALUES OF SINE, COSINE AND TANGENT OF THI

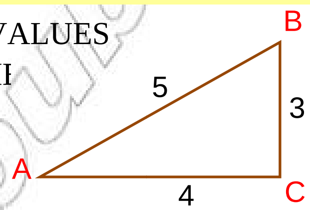


Figure 5.75

SOLUTION:

$\text{SIN A} = \frac{\text{OPP}}{\text{HYP}} = \frac{\text{BC}}{\text{AB}} = \frac{3}{5};$　　$\text{COS A} = \frac{\text{ADJ}}{\text{HYP}} = \frac{\text{AC}}{\text{AB}} = \frac{4}{5};$　　$\text{TAN A} = \frac{\text{OPP}}{\text{ADJ}} = \frac{\text{BC}}{\text{AC}} = \frac{3}{4}$

SIMILARLY, $\text{SIN B} = \frac{\text{OPP}}{\text{HYP}} = \frac{\text{AC}}{\text{AB}} = \frac{4}{5};$　　$\text{COS B} = \frac{\text{ADJ}}{\text{HYP}} = \frac{\text{BC}}{\text{AB}} = \frac{3}{5}$

$\text{TAN B} = \frac{\text{OPP}}{\text{ADJ}} = \frac{\text{AC}}{\text{BC}} = \frac{4}{3}$

ACTIVITY 5.9

1 USING RULER AND CES, DRAW AN EQUILATERAL TRIAN WHICH EACH SID CM LON DRAW THE ALTITUDE $\overline{AD}$ PERPENDICULAR $\overline{BC}$.

A WHAT IS M(∠ABD)? M(∠BAD) ? GIVE REASONS.

B FIND THE LENGTHS I AD (WRITE THE ANSWERS IN SIMPLIFI RADICAL FORM).

C USE THESE TO FIND°, TAN 30°, COS 30°, SIN 60°, TAN 60°, COS 60°. WHAT DO YOU NOTICE

2 DRAW AN ISOSCELES TRIANGLE ABC IN WHICH ∠C IS A RIGHT ANGLE AND AC = 2 CM.

- **A** WHAT IS M(∠A)?
- **B** CALCULATE THE LENGTHS AB AND BC (LEAVE YOUR ANSWER IN RADICAL FORM).
- **C** CALCULATE SIN 45°, COS 45°, TAN 45°.

FROM THE ABOVE ACTIVITY, YOU HAVE PROBABLY DISCOVERED THAT THE VALUES OF SINE, COSINE AND TANGENT OF THE ANGLES 30°, 45° AND 60° ARE AS SUMMARIZED IN THE FOLLOWING TABLE.

∠A	30^o	45^o	60^o
SIN A	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
COS A	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
TAN A	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

THE ANGLES 30°, 45° AND 60° ARE CALLED *special angles*, BECAUSE THEY HAVE THESE EXACT TRIGONOMETRIC RATIOS.

EXAMPLE 2 A LADDER 6 M LONG LEANS AGAINST A WALL AND MAKES AN ANGLE OF 60° WITH THE GROUND. FIND THE HEIGHT OF THE WALL. HOW FAR FROM THE WALL IS THE FOOT OF THE LADDER?

SOLUTION: CONSIDER △ABC IN THE FIGURE.

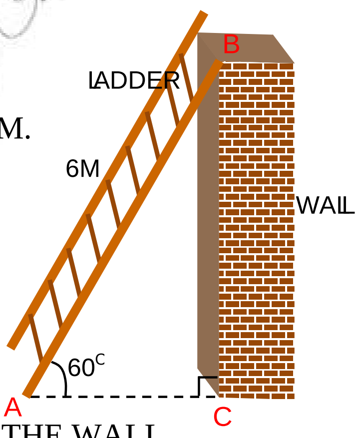


Figure 5.76

M(∠A) = 60°, M(∠C) = 90°, M(∠B) = 30° AND AB = 6 M.

WE WANT TO FIND BC AND AC.

TO FIND BC, WE USE SIN 60° = $\frac{BC}{AB}$. BUT, SIN 60° = $\frac{\sqrt{3}}{2}$

SO, $\frac{\sqrt{3}}{2} = \frac{BC}{6}$

THEREFORE, BC = $3\sqrt{3}$ M, WHICH IS THE HEIGHT OF THE WALL.

TO FIND THE DISTANCE BETWEEN THE FOOT OF THE LADDER AND THE WALL, WE USE

COS 60° = $\frac{AC}{AB}$. COS 60° = $\frac{1}{2}$ AND AB = 6.

SO, $\frac{1}{2} = \frac{AC}{6}$ WHICH IMPLIES AC = 3 M.

In the above example, if the angle that the ladder made with the ground were 50°, how would you solve the problem?

TO SOLVE THIS PROBLEM, YOU WOULD NEED *trigonometric tables*, WHICH GIVE YOU THE VALUES OF SIN 50° AND COS 50°.

5.3.3 Trigonometrical Values of Angles from Tables

(*sin , cos and tan , for* $0^o \leq$ $< 180^o$)

IN THE PREVIOUS SEWTIONE, CREATED A TABLE OF TRIGONOMETRIC RATIOS FOR THE SPECIAL (NAMELY 30^o, 45^o AND 60^o). THEORETICALLY, FOLLOWING THE SAME, A TABLE OF TRIGONOMETRIC RATIOS CAN BE CONSTRUCTED. THERE ARE TABLES OF APP VALUES OF TRIGONOMETRIC RATIOS OF A HAVE ALREADY CONSTRUCTED BY ADVANCED ARITHMETICAL PROCESSES. ONE INCLUDED AT THE END OF THIS BOOK.

ACTIVITY 5.10

USING THE TRIGONOMETRIC, FIND THE VALUE OF EACH OF THE

A COS 50^o **B** SIN 2^o **C** TAN 10^o **D** SIN 80^o

IF YOU KNOW THE VALUE OF ONE OF THE RATIOS OF AN ANGLE, YOU CAN USE OF TRIGONOMETRIC RATIOS TO FIND THE ANGLE. THE PROCEDURE IS ILLUSTR EXAMPLE.

EXAMPLE 1 FIND THE MEASURE OF THE ACUT, CORRECT TO THE NEAR, IF SIN A^o = 0.521.

SOLUTION: REFERRING TO THE "SINE" COLUMN TABLE, WE FIND THAT 0.521 DO APPEAR THERE. THE TWO VALUES IN THE TABLE CLOSEST TO ONE LARGER) ARE 0.515 AND 0.530. THESE VALUES TO 31^o AND 32^o, RESPECTIVELY.

NOTE THAT 0.521 IS CLOSER, WHOSE VALUE CORRESPONDS TO 31^o

THEREFORE, m(∠A) = 31^o (*to the nearest degree*)

ACTIVITY 5.11

1 USE YOUR TRIGONOMETRIC FIND THE VALUE OF THE ACUT CORRECT TO THE NEAR.

A SIN (A)= 0.92 **D** SIN (A) = 0.981

B COS (A) = 0.984 **E** COS (A) = 0.422

C TAN (A) = 0.38 **F** TAN (A) = 2.410

2 USE YOUR CALCULATOR TO FIN (check your calculator is in degrees mode)

USING TRIGONOMETRIC RATIOS, YOU CAN N-ANGLED TRIANGLES AN PROBLEMS. TO SOLVE A RIGHT-ANGLED TRIANGLE MEANS TO FIND THE MISSING PARTS OF THE WHEN SOME PARTS ARE GIVEN. FOR EXAMPLE, IF YOU ARE GIVEN THE LE MEASURE OF AN ANGLE (OTHER THAN THE , YOU CAN USE THE APPI TRIGONOMETRIC RATIO TO FIND THE REQUIRE

IN SHORT, IN SOLVING A RIGHT-ANGLED TRIANGLE, WE NEED TO USE

A THE TRIGONOMETRIC RATIOS OF

B Pythagoras theorem WHICH IS $a^2 + b^2 = c^2$, WHERE a IS THE LENGTH OF T OPPOSITE TO $\angle A$, b IS THE LENGTH OF THE SIDE OPPOSITE TO $\angle B$ AND c IS THE LENGTH OF THE hypotenuse.

EXAMPLE 2 FIND THE LENGTHS OF THE SIDES INDICATED BY

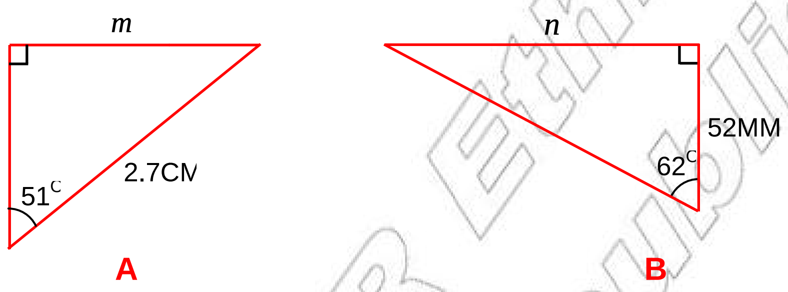


Figure 5.77

SOLUTION:

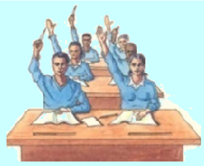
A $\sin 51° = \frac{m}{2.7}$.

SO, $m = 2.7 \sin 51° = 2.7 \times 0.777 \approx 2.1$ CM (*1 decimal place*)

B $\tan 62° = \frac{n}{52}$.

SO, $n = 52 \tan 62° = 52 \times 1.881 \approx 98$ MM (*to the nearest mm*)

ACTIVITY 5.12



USING FIGURE 5.78, WRITE EACH OF THE FC IN TERMS OF THE LENG c.

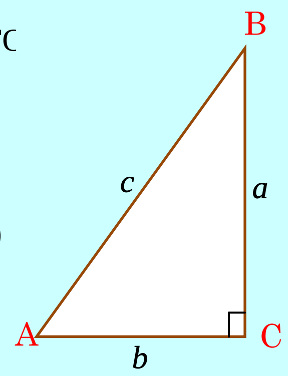


Figure 5.78

1 **A** $\sin(\angle A)$ **B** $\cos(\angle A)$

C $\tan(\angle A)$ **D** $\frac{\sin(\angle A)}{\cos(\angle A)}$

E $\sin(\angle B)$ **F** $\cos(\angle B)$

G $\tan(\angle B)$ **H** $\frac{\sin(\angle B)}{\cos(\angle B)}$

2 **A** $(\text{SIN}(\angle A))^2$

B $(\text{COS}(\angle A))^2$

C WRITE THE VALUE OF $\text{SIN}^2(\angle A) + \text{COS}^2(\angle A)$.

Notation: WE ABBREVIATE $(\text{SIN}(\angle A))^2$ AS $\text{SIN}^2(\angle A)$. SIMILARLY, WE WRITE $\text{COS}^2(\angle A)$ AND $\text{TAN}^2(\angle A)$ INSTEAD OF $(\text{COS}(\angle A))^2$ AND $(\text{TAN}(\angle A))^2$, RESPECTIVELY.

DO YOU NOTICE ANY INTERESTING RESULTS FROM ACTIVITY? STATE THEM.

YOU MIGHT HAVE DISCOVERED THAT

1 IF $m(\angle A) + m(\angle B) = 90^o$, I.E., A AND B ARE COMPLEMENTARY ANGLES, THEN

I $\text{SIN}(\angle A) = \text{COS}(\angle B)$ **II** $\text{COS}(\angle A) = \text{SIN}(\angle B)$

2 $\text{TAN}(\angle A) = \dfrac{\text{SIN}(\angle A)}{\text{COS}(\angle A)}$

3 $\text{SIN}^2(\angle A) + \text{COS}^2(\angle A) = 1$

How can you use the trigonometric table to find the sine, cosine and tangent of obtuse angles such as 95°, 129°, and 175°?

SUCH ANGLES ARE NOT LISTED IN THE TABLE.

BEFORE WE CONSIDER HOW TO FIND THE TRIGONOMETRIC RATIO OF OBTUSE ANG REDEFINE THE TRIGONOMETRIC RATIOS BY USING DIRECTED DISTANCE. TO DO THIS, W RIGHT ANGLE TRIANGLE POA AS GIVEN IN FIGURE 5.79. ANGLE POA IS THE ANTICLOCKWISE ANGLE FROM THE POSITIVE x-AXIS.

NOTE THAT THE LENGTHS OF THE SIDES CAN BE EXPRESSED IN TERMS OF THE COORDINA

I.E., $OA = x$, $AP = y$, AND USING PYTHAGORAS THEOREM, WE HAVE,

$$OP = \sqrt{x^2 + y^2}$$

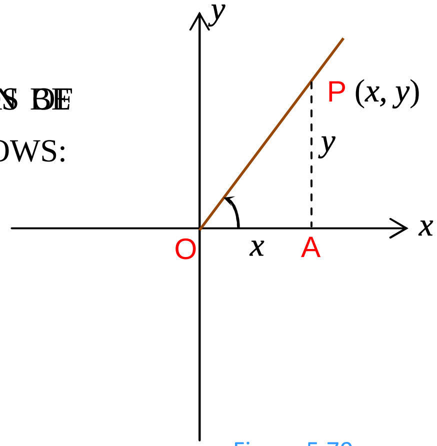


Figure 5.79

AS A RESULT, THE TRIGONOMETRIC RATIOS OF $\angle POA$ CAN BE EXPRESSED IN TERMS OF x, y AND $\sqrt{x^2 + y^2}$, AS FOLLOWS:

$$\text{SIN}(\angle POA) = \frac{\text{OPP}}{\text{HYP}} = \frac{AP}{OP} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\text{COS}(\angle POA) = \frac{\text{ADJ}}{\text{HYP}} = \frac{OA}{OP} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\text{TAN}(\angle POA) = \frac{\text{OPP}}{\text{ADJ}} = \frac{AP}{OA} = \frac{y}{x}$$

I.E., $\text{SIN}(\angle POA) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $\text{COS}(\angle POA) = \dfrac{x}{\sqrt{x^2 + y^2}}$; $\text{TAN}(\angle POA) = \dfrac{y}{x}$

FROM THE ABOVE DISCUSSION, IT IS POSSIBLE TO COMPUTE THE VALUES OF TRIGONO USING ANY POINT ON THE TERMINAL SIDE OF THE ANGLE.

LET US NOW FIND THE SINE AND COSINE OF 129^o USING THE TABLE. TO DO THIS, WE FIRST PUT 129^o ON THE xy-PLANE, SO THAT ITS VERTEX IS AT THE ORIGIN AND ITS INITIAL SIDE ON THE POSITIVE x-AXIS.

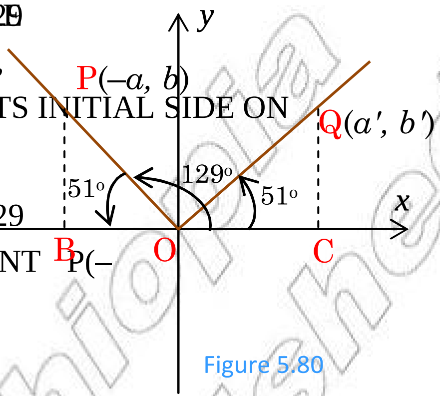


Figure 5.80

I TO FIND SIN 129^o, WE FIRST EXPRESS SIN 129^o IN TERMS OF THE COORDINATES OF THE POINT P($-a$, b). SO, WE HAVE,

$$\text{SIN } 129^o = \frac{b}{\sqrt{a^2+b^2}}.$$

WHAT ACUTE ANGLE PUT IN THE xy-PLANE HAS THE SAME y VALUE (THAT IS b)?

IF WE DRAW THE $\angle COQ = 51^o$ SO THAT OP = OQ, THEN WE SEE THAT

$\Delta BOP \cong \Delta COQ$. SO WE HAVE

BP = CQ AND OB = OC

IT FOLLOWS THAT SIN 129^o = SIN 51^o. FROM THE TABLE SIN 51^o = 0.777.

HENCE, SIN 129^o = 0.777

NOTICE THAT SIN 129^o = SIN ($180^o - 129^o$)

THIS CAN BE GENERALIZED AS FOLLOWS.

IF θ IS AN OBTUSE ANGLE, I.E., $90^o < \theta < 180^o$, THEN

$$\text{SIN } \theta = \text{SIN } (180 - \theta) \qquad \text{I}$$

II TO FIND COS 129^o

HERE ALSO WE FIRST EXPRESS COS 129^o IN TERMS OF THE COORDINATES OF P ($-a$, b). SO,

$$\text{COS } 129^o = \frac{-a}{\sqrt{a^2+b^2}}.$$

BY TAKING $180^o - 129^o$, WE FIND THE ACUTE ANGLE 51^o.

SINCE $\Delta BOP \cong \Delta COQ$, WE SEE THAT OC = OB, BUT IN THE OPPOSITE DIRECTION. SO, TH x VALUE OF P IS THE OPPOSITE OF THE x VALUE OF Q. THAT IS $-a = a'$

THEREFORE, $\cos 129^o = \frac{a'}{\sqrt{a^2+b^2}} = -\cos 51^o$

FROM THE TRIGONOMETRIC TABLE, YOU HAVE COS 51° = 0.629

THEREFORE, COS 129° = –0.629.

THIS DISCUSSION LEADS YOU TO THE FOLLOWING GENERALIZATION.

IF IS AN OBTUSE ANGLE, THEN

COS = – COS (180° –) II

EXAMPLE 3 WITH THE HELP OF THE TRIGONOMETRIC TABLE, FIND THE APPROXIMATE VALUES OF:

A COS 100° **B** SIN 163° **C** TAN 160°

SOLUTION: A USING THE RULE COS = – COS (180° –), WE OBTAIN;

COS 100° = – COS (180° – 100°) = – COS 80°

FROM THE TRIGONOMETRIC TABLE, WE HAVE COS 80° = 0.174.

THEREFORE, COS 100° = –0.174.

B FROM THE RELATION SIN = SIN (180 –), WE HAVE

SIN 163° = SIN (180° – 163°) = SIN 17°

FROM THE TABLE SIN 17° = 0.292

THEREFORE, SIN 163° = 0.292

C TO FIND TAN 160°,

$$\tan 160^o = \frac{\sin 160^o}{\cos 160^o} = \frac{\sin 20^o}{-\cos 20^o} = -\left(\frac{\sin 20^o}{\cos 20^o}\right) = -\tan 20^o$$

FROM THE TABLE, WE HAVE TAN 20° = 0.364.

THEREFORE, TAN 160° = –0.364.

TO SUMMARIZE, FOR A POSITIVE OBTUSE ANGLE ,

SIN = SIN (180° –)

COS = – COS (180° –)

TAN = – TAN (180° –)

Exercise 5.9

1 **I** EXPRESS EACH OF THE FOLLOWING RADIAN MEASURES IN DEGREES.

A $\frac{\;}{6}$ **B** **C** $\frac{\;}{3}$, **D** 2 **E** $\frac{3}{4}$ **F** 5

II EXPRESS EACH OF THE FOLLOWING IN RADIAN MEASURE:

A 270^o **B** 150^o **C** 225^o **D** 15^o

2 WITHOUT USING A TABLE, FIND THE VALUE OF EACH OF THE FOLLOWING. (ANSWERS MAY BE LEFT IN RADICAL FORM.)

A SIN $\frac{\;}{6}$ **B** TAN $\frac{3}{4}$ **C** COS 150^o **D** TAN $\frac{2}{3}$

3 IN ΔABC, IF M($\angle A$) = 53^o, AC = 8.3 CM AND M($\angle C$) = 90^o, FIND BC, CORRECT TO THE NEAREST WHOLE NUMBER.

4 A LADDER 20 FT LONG LEANS AGAINST A BUILDING, MAKING AN ANGLE OF 61^o WITH THE GROUND. DETERMINE, CORRECT TO THE NEAREST FT, HOW FAR UP THE BUILDING THE LADDER REACHES.

5 EXPRESS EACH OF THE FOLLOWING IN TERMS OF THE SINE OR COSINE OF AN ACUTE ANGLE:

A COS 165^o **B** SIN 126^o **C** COS $\frac{3}{5}$ **D** SIN 139^o

6 IN EACH OF THE FOLLOWING, FIND THE LENGTH OF THE HYPOTENUSE (a).

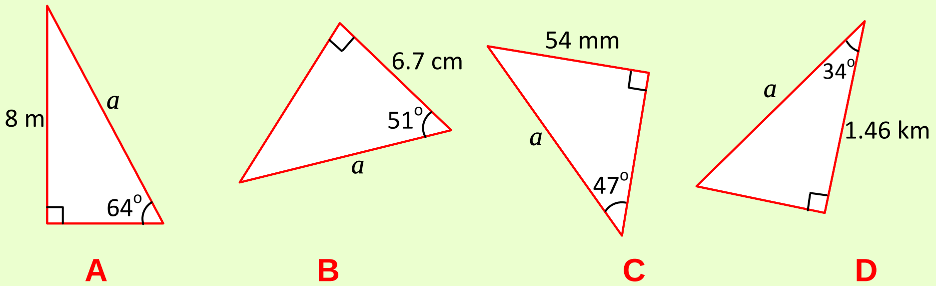


Figure 5.81

7 FIND THE SINE, COSINE AND TANGENT OF EACH OF THE FOLLOWING ANGLES FROM THE TABLE.

A 25^o **B** 63^o **C** 89^o

D 135^o **E** 142^o **F** 173^o

8 USE THE TRIGONOMETRIC TABLE INCLUDED AT THE END OF THE BOOK TO FIND THE DEGREE MEASURE OF $\angle P$ IF:

A SIN P = 0.83 **B** COS P = 0.462 **C** TAN P = 0.945

D SIN P = $\frac{1}{4}$ **E** COS P = 0.824

5.4 CIRCLES

5.4.1 Symmetrical Properties of Circles

ACTIVITY 5.13

1 WHAT IS A CIRCLE?

2 WHAT IS A LINE OF SYMMETRY?

3 WHICH OF THE FOLLOWING FIGURES HAVE A LINE OF SYMMETRY?

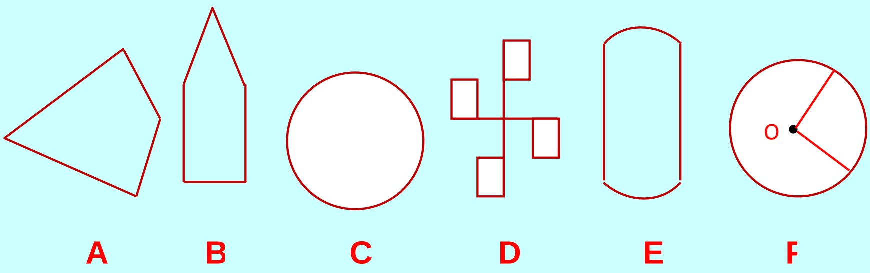


Figure 5.82

RECALL THAT A CIRCLE IS DEFINED AS THE SET OF POINTS IN A GIVEN PLANE, EACH OF WHICH IS THE SAME DISTANCE FROM A FIXED POINT OF THE PLANE. THE FIXED POINT IS CALLED THE centre, AND THE DISTANCE IS THE radius OF THE CIRCLE.

A LINE SEGMENT THROUGH THE CENTRE OF A CIRCLE WITH END POINTS ON THE CIRCLE IS CALLED A diameter. A chord OF A CIRCLE IS A LINE SEGMENT WHOSE END POINTS LIE ON THE CIRCLE.

IN SECTION 5.1.2, YOU LEARNED THAT IF ONE PART OF A FIGURE CAN BE MADE TO FIT EXACTLY ON THE REST OF THE FIGURE BY FOLDING IT ABOUT $\overleftrightarrow{AB}$, THE FIGURE IS SAID TO BE SYMMETRICAL ABOUT $\overleftrightarrow{AB}$, AND THE STRAIGHT LINE $\overleftrightarrow{AB}$ IS CALLED THE line of symmetry. FOR EXAMPLE, EACH OF THE FOLLOWING FIGURES IS SYMMETRICAL ABOUT THE LINE $\overleftrightarrow{AB}$.

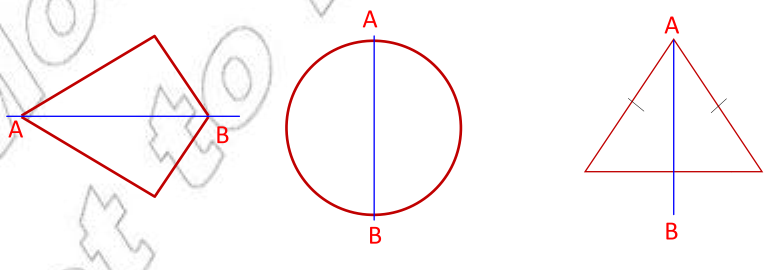


Figure 5.83

OBSERVE THAT IN A SYMMETRICAL FIGURE THE LENGTH OF ANY LINE SEGMENT OR T
ANGLE IN ONE HALF OF THE FIGURE IS EQUAL TO THE LENGTH OF THE CORRESPONDING
THE SIZE OF THE CORRESPONDING ANGLE IN THE OTHER HALF OF THE FIGURE.

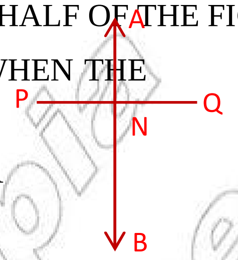


Figure 5.84

IF IN THE FIGURE ON THE RIGHT, P COINCIDES WITH Q WHEN THE FIGURE IS FOLDED ABOUT $\overleftrightarrow{AB}$ AND IF $\overline{PQ}$ INTERSECTS $\overleftrightarrow{AB}$ AT N THEN ∠PNA COINCIDES WITH ∠QNA AND THEREFORE EACH IS A RIGHT ANGLE AND PN = QN.

THEREFORE,

IF P AND Q ARE CORRESPONDING POINTS FOR A LINE OF SYMMETRY $\overleftrightarrow{AB}$, THEN THE PERPENDICULAR BISECTOR OF $\overline{PQ}$ IS $\overleftrightarrow{AB}$. CONVERSELY, IF $\overleftrightarrow{AB}$ IS THE PERPENDICULAR BISECTOR OF $\overline{PQ}$, THEN P AND Q ARE CORRESPONDING POINTS FOR THE LINE OF SYMMETRY $\overleftrightarrow{AB}$ AND WE SAY THAT Q IS THE IMAGE OF P IN $\overleftrightarrow{AB}$ AND P IS THE IMAGE OF Q IN $\overleftrightarrow{AB}$.

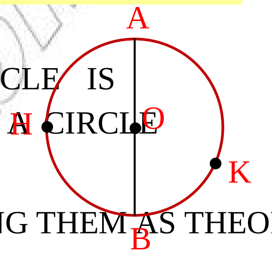


Figure 5.85

IN THE ADJACENT FIGURE, O IS THE CENTRE AND $\overline{AB}$ IS A DIAMETER OF THE CIRCLE. NOTE THAT A CIRCLE IS SYMMETRICAL ABOUT ITS DIAMETER. THEREFORE, A CIRCLE HAS AN INFINITE NUMBER OF LINES OF SYMMETRY.

WE NOW DISCUSS SOME PROPERTIES OF A CIRCLE, STATING THEM AS THEOREMS.

Theorem 5.9

The line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

Proof:-

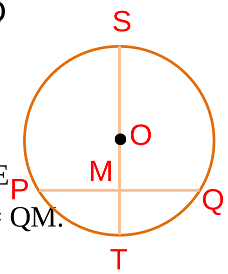


Figure 5.86

GIVEN: A CIRCLE WITH CENTRE O AND A CHORD $\overline{PQ}$ WHOSE MIDPOINT IS M.

WE WANT TO PROVE THAT ∠OMP IS A RIGHT ANGLE.

CONSTRUCTION: DRAW THE DIAMETER ST THROUGH M. THE CIRCLE IS SYMMETRICAL ABOUT THE LINE ST. BUT PM = QM.

SO, ST IS THE PERPENDICULAR BISECTOR OF PQ.

THIS COMPLETES THE PROOF.

Theorem 5.10

The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord.

Proof:-

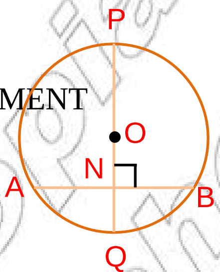


Figure 5.87

GIVEN: A CIRCLE WCENTRE O, AND THE LINE SEGMENT ON DRAWN FROM O PERPENDICULAR TAB AS SHOWN IN THE ADJACEI

WE WANO TPROVE THAT AN

CONSTRUCTION: DRAW TDIAMETER PQ THROUGH N.

THEN THECIRCLE IS SYMMETRIC.PQ. BUT PQ⊥ AB AND A AND B ARE ON THI

THEREFORE, PQ IS THE PERPENDICULAR B

ACTIVITY 5.14

1 PROVETHEOREM 5.1AND5.11 USING CONGRUENCY OF TRIA

2 A CHORD OF LEN(CM IS AT A DISTANCE OF 12 CM FR CENTRE OF A CIRFIND THE RA OF THE CIRCLE.

3 A CHORD OF A CIRCLE OF CM IS 8 CM LONFIND THE DISTANCE OF TI FROM THE CENTR

4 $\overline{AB}$ AND $\overline{CD}$ ARE EQUAL CHORDS IN A CIRCL CM. IF EACH CHOR CM, FIND THEIR DISTANCE FROM THE CENT

5 DEFINE WHAT YOU MI'*a line tangent to a circle*'.

6 HOW MANY TANGENTS ARE THERE FROM AN EXTERNAL POINT TO COMPARE THEIRG.

SOME OTHER PROPERTIES OF CAN ALSO BE PROVED BY USING THE FACT THAT A (SYMMETRICAL ABOUT ANY DIAMETER

Theorem 5.11

i If two chords of a circle are equal, then they are equidistant from the centre.

ii If two chords of a circle are equidistant from the centre, then their lengths are equal.

Theorem 5.12

If two tangent segments are drawn to a circle from an external point, then,

i the tangents are equal in length, and

ii the line segment joining the centre to the external point bisects the angle between the tangents.

Restatement: IF TP IS A TANGENT TO A CIRCLE AT P WHOSE CENTRE IS O AND TQ IS ANOTHER TANGENT TO THIS CIRCLE AT Q, THEN,

I TP = TQ   II M(∠OTP) = M(∠OTQ)

Proof:-

I ΔOTP AND ΔOTQ ARE RIGHT ANGLED TRIANGLES WITH RIGHT ANGLES AT P AND

(A radius is perpendicular to a tangent at the point of tangency).

II OBVIOUSLY OT = OT

AND OP = OQ (*why?*)

III ∴ ΔOTP ≅ ΔOTQ (*why?*)

SO, TP = TQ AND M(∠OTP) = M(∠OTQ), AS REQUIRED.

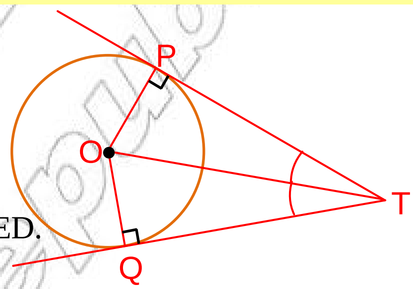


Figure 5.88

5.4.2 Angle Properties of Circles

WE START THIS SUBSECTION BY A REVIEW AND DISCUSSION OF SOME IMPORTANT TER
TO THE DIAGRAMS IN FIGURE 5.89 WILL HELP YOU TO UNDERSTAND SOME OF THESE TERMINOL
(IN EACH CIRCLE, O IS THE CENTRE.)

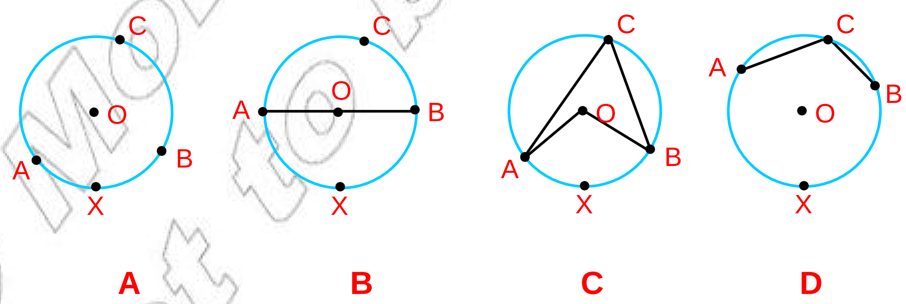


Figure 5.89

- ✓ A PART OF A CIRCLE (PART OF ITS CIRCUMFERENCE) BETWEEN ANY TWO POINTS ON THE CIRCLE, SAY BETWEEN A AND B, IS CALLED AN arc AND IS DENOTED BY $\overset{\frown}{AB}$. HOWEVER, THIS NOTATION CAN BE AMBIGUOUS SINCE THERE ARE TWO ARCS OF THE CIRCLE WIT END POINTS. THEREFORE, WE EITHER USE THE TERMS MINOR ARC AND MAJOR ARC ANOTHER POINT, SAY X, ON THE DESIRED ARC AND THEN USE THE NOTATION $\overset{\frown}{AXB}$. FOR EXAMPLE, IN FIGURE 5.89A, $\overset{\frown}{AXB}$ IS THE PART OF THE CIRCLE WITH A AND B AS ITS EN POINTS AND CONTAINING THE POINT X. THE REMAINING PART OF THE CIRCLE, I.E WHOSE END POINTS ARE A AND B BUT CONTAINING C IS THE ARC $\overset{\frown}{ACB}$.
- ✓ IF AB IS A DIAMETER OF A CIRCLE (SEE FIGURE 5.89B), THEN THE ARC $\overset{\frown}{ACB}$ $\left(\text{OR } \overset{\frown}{AXB}\right)$ IS CALLED A semicircle. NOTICE THAT A SEMICIRCLE IS HALF OF THE CIRCUMFERENCE OF THE CIRCLE. AN ARC IS SAID TO BE A minor arc, IF IT IS LESS THAN A SEMICIRCLE AND A major arc, IF IT IS GREATER THAN A SEMICIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, $\overset{\frown}{AXB}$ IS MINOR ARC WHILE $\overset{\frown}{ACB}$ IS A MAJOR ARC.

A central angle OF A CIRCLE IS AN ANGLE WHOSE VERTEX IS AT THE CENTRE OF THE CIRCLE AND WHOSE SIDES ARE RADII OF THE CIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, THE ANGLE ∠AOB IS A CENTRAL ANGLE. IN THIS CASE, WE SAY THAT ∠AOB IS subtended BY THE ARC $\overset{\frown}{AXB}$ (OR BY THE CHORD AB). HERE, WE MAY ALSO SAY THAT THE ANGLE ∠AOB intercepts THE ARC $\overset{\frown}{AXB}$.

RECALL THAT THE MEASURE OF A CENTRAL ANGLE EQUALS THE ANGLE MEASURE OF TH THUS, IN FIGURE 5.89C,

$$m(\angle AOB) = m\left(\overset{\frown}{AXB}\right).$$

AN inscribed angle IN A CIRCLE IS AN ANGLE WHOSE VERTEX IS ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, ∠ACB IS AN INSCRIBED ANGLE. HERE ALSO, THE INSCRIBED ANGLE ∠ACB IS SAID TO BE subtended BY THE ARC $\overset{\frown}{AXB}$ (OR BY THE CHORD $\overline{AB}$).

- ✓ OBSERVE THAT THE VERTEX OF AN INSCRIBED ANGLE ∠ACB IS ON THE ARC $\overset{\frown}{ACB}$. THIS ARC, $\overset{\frown}{ACB}$, CAN BE A SEMICIRCLE, A MAJOR ARC OR A MINOR ARC. IN SUCH CASES, WE THAT THE ANGLE ∠ACB IS INSCRIBED IN A SEMICIRCLE, MAJOR ARC OR MINOR A RESPECTIVELY. FOR EXAMPLE, IN FIGURE 5.89B, ∠ACB IS INSCRIBED IN A SEMICIRCLE, IN FIGURE 5.89C ∠ACB IS INSCRIBED IN THE MAJOR ARC, AND IN FIGURE 5.89D ∠ACB IS INSCRIBED IN THE MINOR ARC.

Theorem 5.13

The measure of a central angle subtended by an arc is twice the measure of an inscribed angle in the circle subtended by the same arc.

BOTH DRAWINGS IN FIGURE 5.90 ILLUSTRATE THIS THEOREM.

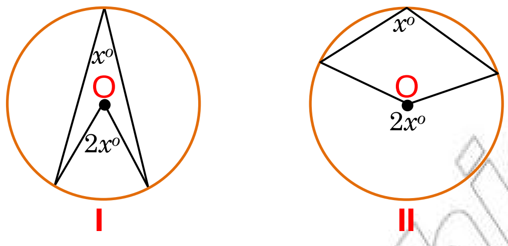


Figure 5.90

Exercise 5.10

IN EACH OF THE FOLLOWING FIGURES, O IS THE CENTRE OF THE CIRCLE. CALCULATE T
THE ANGLES MARKED x.

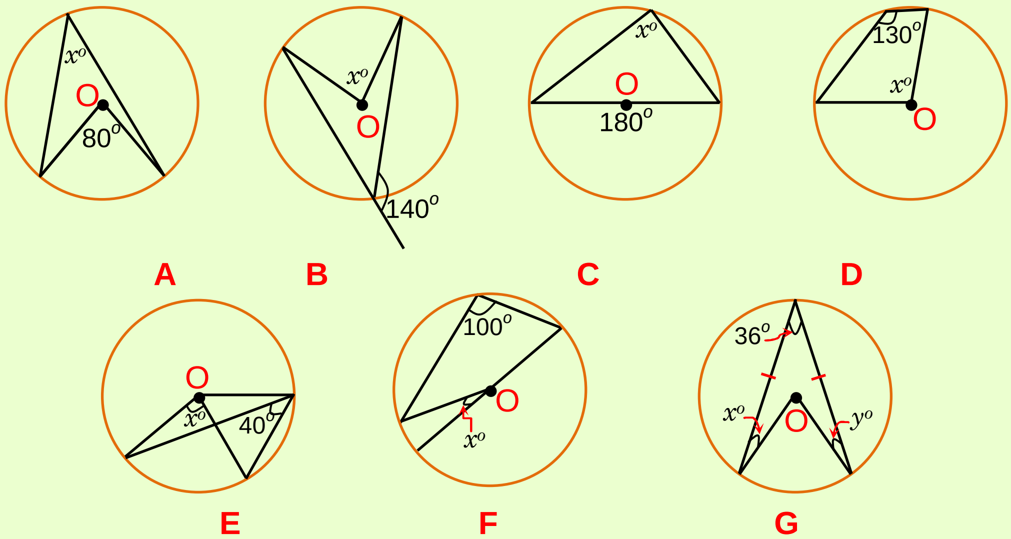


Figure 5.91

Corollary 5.13.1

Angles inscribed in the same arc of a circle (*i.e., subtended by the same arc*) are equal.

Proof:-

BY THE ABOVE THEOREM, EACH OF THE ANGLES INSCRIBED IN THE CIRCLE SUBTENDED BY THE ARC IS EQUAL TO HALF OF THE CENTRAL ANGLE SUBTENDED BY THE ARC. HENCE, THEY ARE EQUAL

Corollary 5.13.2 Angle in a semicircle

The angle inscribed in a semi-circle is a right angle.

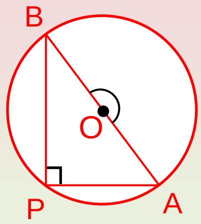


Figure 5.92

Proof:-

THE GIVEN ANGLE, ∠APB, IS SUBTENDED BY A SEMICIRCLE. THE CORRESPONDING CENTRAL ANGLE SUBTENDED BY $\overline{AB}$ IS STRAIGHT ANGLE. I.E., THE CENTRAL ANGLE IS 180^o. HENCE, BY

THEOREM 5.14 $M(\angle APB) = \frac{1}{2} M(\angle AOB) = \frac{1}{2} \times 180^O = 90^O$.

THIS COMPLETES THE PROOF.

Exercise 5.11

1 CALCULATE THE MARKED ANGLES IN EACH OF THE FOLLOWING FIGURES.

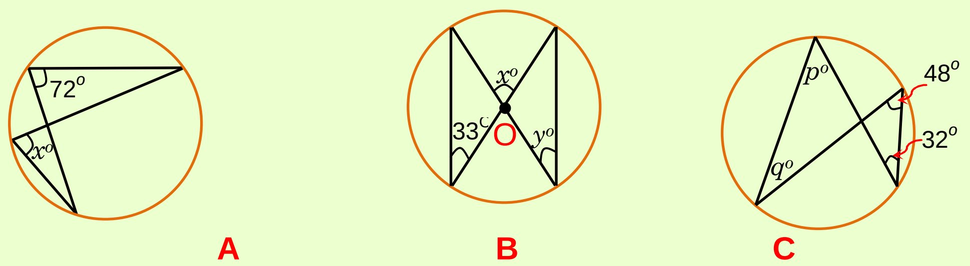


Figure 5.93

2 IN EACH OF THE FOLLOWING FIGURES, O IS THE CENTRE AND $\overline{AB}$ IS THE DIAMETER OF THE CIRCLE. CALCULATE THE VALUE OF x IN EACH CASE.

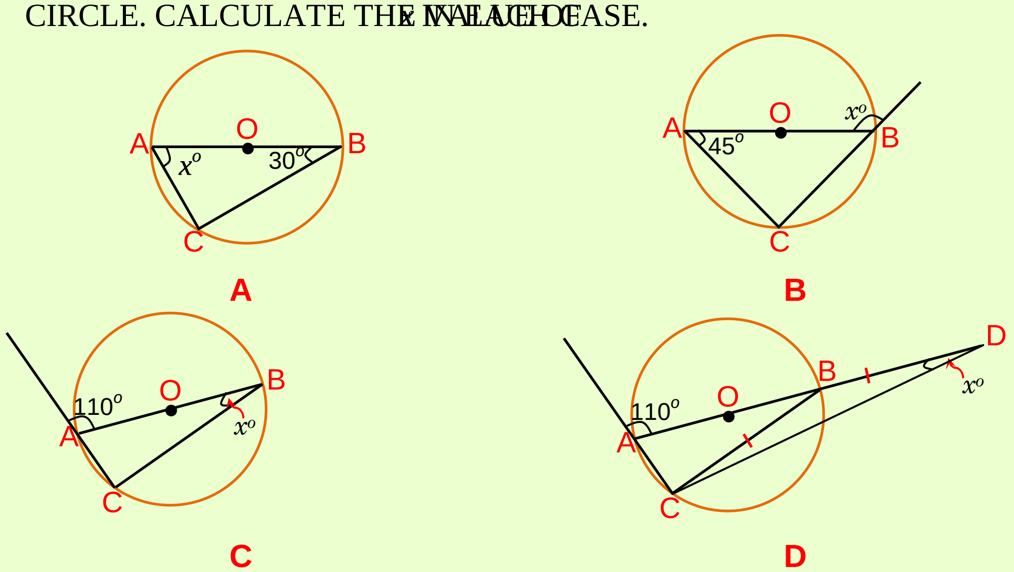


Figure 5.94

Corollary 5.13.3

Points P, Q, R and S all lie on a circle. They are called concyclic points.

Joining the points P, Q, R and S produces a cyclic quadrilateral.

The opposite angles of a cyclic quadrilateral are supplementary. i.e.,

$$m(\angle P) + m(\angle R) = 180^\circ \text{ and } m(\angle S) + m(\angle Q) = 180^\circ.$$

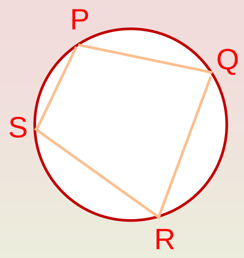


Figure 5.95

ACTIVITY 5.15

CALCULATE THE LANGLES IN EACH OF THE]:

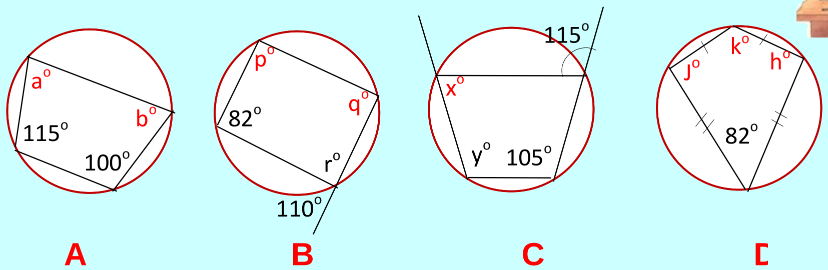


Figure 5.96

5.4.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

Remember that:

- ✓ CIRCUMFERENCE OF A C r OR d.
- ✓ AREA OF A CIRCL r^2.
- ✓ PART OF THE CIRCUMFERENCE OF A CIRarc.
- ✓ A segment OF A CIRCLE IS A REGION BOUNDED BY A C.
- ✓ A sector OF A CIRCLE IS BOUNDED BY TWO RA.

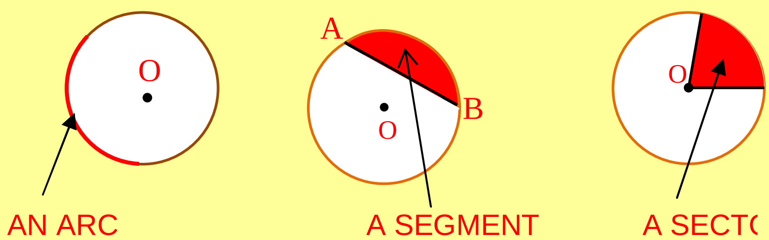


Figure 5.97

Group Work 5.5

- WHAT FRACTION OF A COMPLETE CIRCLE IS THE SHADED REGION IN FIGURE 5.98?
- WHAT IS THE AREA OF THE SHADED REGION IN THIS FIGURE?

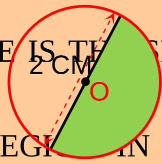


Figure 5.98

- WHAT IS THE AREA IF THE SHADED REGION IS A QUADRANT (FIGURE 5.99)?

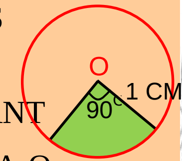


Figure 5.99

- WHAT IS THE AREA IF THE SHADED REGION IS PART OF A QUADRANT BOUNDED BY A CHORD AND AN ARC AS SHOWN IN FIGURE 5.100?

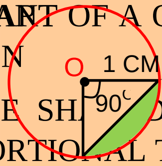


Figure 5.100

DISCUSS HOW TO FIND THE AREA OF EACH OF THE SHADED SECTORS SHOWN BELOW. IS THE AREA OF EACH SECTOR PROPORTIONAL TO THE ANGLE BETWEEN THE RADII BOUNDING THE SECTOR?

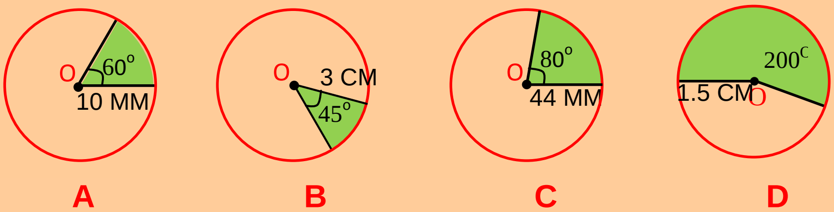


Figure 5.101

DISCUSS HOW TO FIND THE AREA OF EACH OF THE SHADED SEGMENTS SHOWN BELOW:

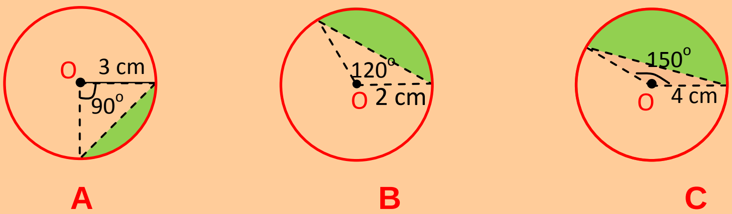


Figure 5.102

✓ Arc length

THE LENGTH ℓ OF AN ARC OF A CIRCLE OF RADIUS r THAT SUBTENDS AN ANGLE OF θ AT THE CENTRE IS GIVEN BY

$$\ell = \frac{\theta}{360^o} \times 2\pi r = \frac{\pi r \theta}{180^o}$$

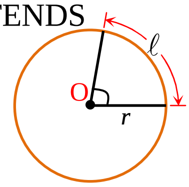


Figure 5.103

✓ The area and perimeter of a sector

THE AREA A OF A SECTOR OF RADIUS r AND CENTRAL ANGLE θ IS GIVEN BY

$$A = \frac{\theta}{360^o} \times \pi r^2 = \frac{\pi r^2 \theta}{360^o}$$

THE PERIMETER P OF THE SECTOR IS THE SUM OF THE RADII AND THE ARC THAT BOUND IT.

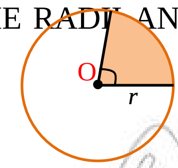


Figure 5.104

$$P = 2r + \frac{r}{180^o}$$

✓ The area and perimeter of a segment

THE AREA A AND PERIMETER P OF A SEGMENT OF A CIRCLE OF RADIUS r CUT OFF BY A CHORD SUBTENDING AN ANGLE AT THE CENTRE OF A CIRCLE ARE GIVEN BY

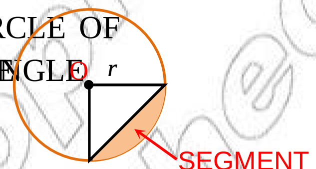


Figure 5.105

$$A = \frac{r^2}{360^o} - \frac{1}{2}r^2 \text{ SIN} \quad (\textit{sector area} - \textit{triangle area})$$

Note: THE AREA FORMULA FOR A TRIANGLE:

$A = \frac{1}{2}ab \text{ SIN}$ WHERE a AND b ARE THE LENGTHS OF ANY TWO SIDES OF THE TRIAN

AND θ IS THE MEASURE OF THE ANGLE INCLUDED BETWEEN THE GIVEN SIDES IS DIS THE NEXT SECTION (SECTION 5.5 OF THIS TEXTBOOK).

$$P = 2r \text{ SIN} \frac{\;}{2} + \frac{r}{180^o} \quad (\textit{chord length} + \textit{arc length})$$

EXAMPLE 1 A SEGMENT OF A CIRCLE OF RADIUS 12 CM IS CUT OFF BY A CHORD SUBTENDING AN ANGLE 60° AT THE CENTRE OF THE CIRCLE. FIND:

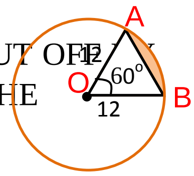


Figure 5.106

A THE AREA OF THE SEGMENT. **B** THE PERIMETER OF THE SECTOR.

SOLUTION:

A FROM THE FIGURE, AREA OF THE SEGMENT (THE SHADED PAR

= AREA OF THE SECTOR OAB – AREA OF TRIANGLE OAB;

AREA OF THE SECTOR OAB = $\frac{\times 12^2 \times 60}{360} = 24$ CM².

AREA OF TRIANGLE OAB = $\frac{1}{2} \times 12^2 \times \text{SIN } 60^o = 36\sqrt{3}$ CM².

THEREFORE, SEGMENT AREA = $(24 - 36\sqrt{3})$ CM².

B PERIMETER OF THE SECTOR = 2 × RADIUS + LENGTH OF ARC AB

$$= 2 \times 12 + \frac{\times 12 \times 60}{180} = (24 + 4\;)\text{ CM}$$

Exercise 5.12

1 CALCULATE THE PERIMETER AND AREA OF EACH OF THE FOLLOWING FIGURES. ALL CURVES ARE SEMICIRCLES OR QUADRANTS.

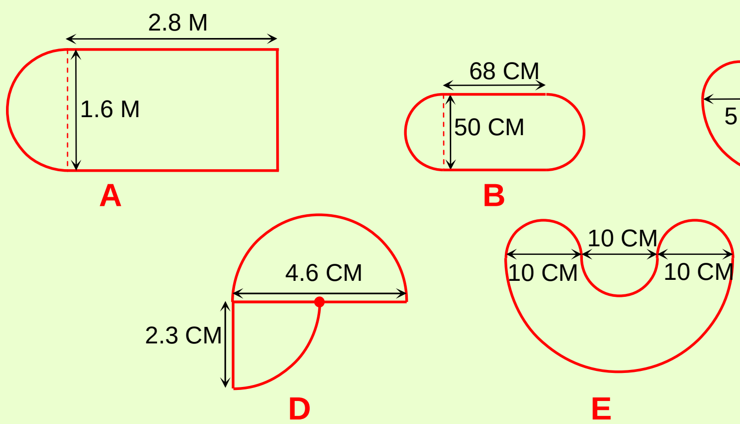


Figure 5.107

2 IN EACH OF THE FOLLOWING SECTORS OPQ FIND:

I THE LENGTH OF ARC PQ.

II AREA OF THE SECTOR OPQ.

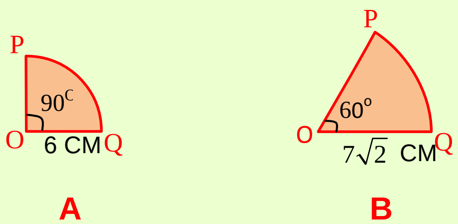


Figure 5.108

3 IN FIGURE 5.109, O IS THE CENTRE OF THE CIRCLE. IF THE RADIUS OF THE CIRCLE IS 4 CM AND $m(\angle AKB) = 30^o$, FIND THE AREA OF THE SEGMENT BOUNDED BY THE CHORD AB AND THE ARC AKB.

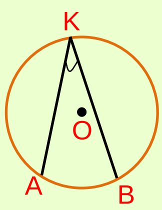


Figure 5.109

4 A SQUARE ABCD IS INSCRIBED IN A CIRCLE OF RADIUS 4 CM. FIND THE AREA OF THE MINOR SEGMENT CUT OFF BY THE CHORD $\overline{AB}$.

5 CALCULATE THE PERIMETER AND AREA OF EACH OF THE FIGURES, WHERE THE CURVES ARE ARCS OF A CIRCLE WITH COMMON CENTRE AT O.

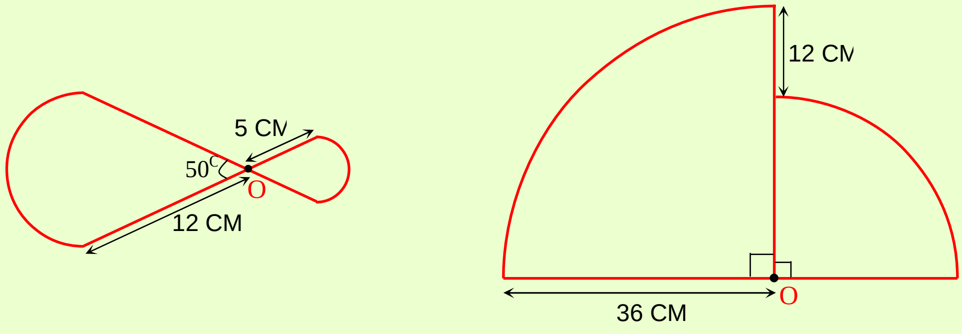


Figure 5.110

5.5 MEASUREMENT

5.5.1 Areas of Triangles and Parallelograms

A Areas of triangles

ACTIVITY 5.16

GIVEN THE RIGHT ANGLED TRIANGLE SHOWN BELOW, VERIFY THAT EACH OF THE FOLLOWING EXPRESSIONS GIVES THE AREA OF △ABC. IN EACH CASE, DISCUSS AND STATE THE FORMULA.

I AREA OF △ABC = $\frac{1}{2}ac$

II AREA OF △ABC = $\frac{1}{2}bh$

III AREA OF △ABC = $\frac{1}{2}bc\sin(\measuredangle A)$

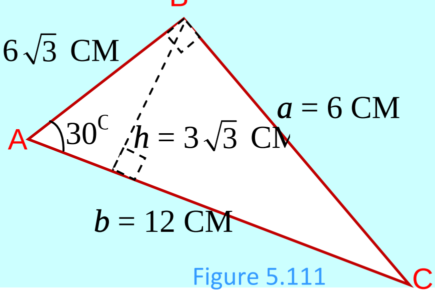


Figure 5.111

THE ABOVE ACTIVITY SHOULD HAVE REMINDED YOU WHAT YOU STUDIED IN LOWER GRADES EXCEPT FOR CASE III, WHICH YOU HAVE USED IN THE PRECEDING SECTION AND GOING TO ABOUT NOW.

CASE I USES THE FOLLOWING:

THE AREA OF A RIGHT ANGLED TRIANGLE WITH PERPENDICULAR SIDES OF LENGTH a AND b IS GIVEN BY

$$A = \frac{1}{2}ab$$

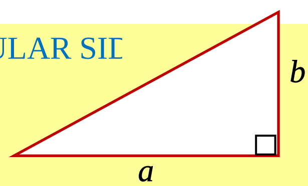


Figure 5.112

CASE II USES THE FOLLOWING FORMULA.

THE AREA A OF ANY TRIANGLE WITH BASE b AND THE CORRESPONDING HEIGHT h IS GIVEN BY

$$A = \frac{1}{2}bh$$

THE BASE AND CORRESPONDING HEIGHT OF A TRIANGLE MAY APPEAR IN ANY ONE OF THE FOLLOWING FORMS.

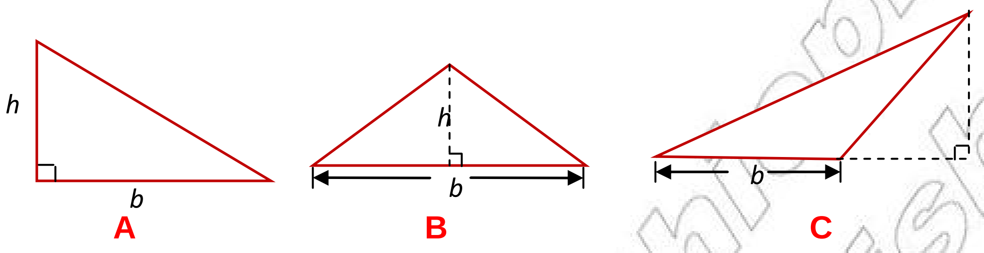


Figure 5.113

FROM THE VERIFICATION OF CASE II, WE COME TO THE FOLLOWING FORMULA.

THE AREA A OF ANY TRIANGLE WITH SIDES a AND b UNITS LONG AND ANGLE C ($\angle$C) INCLUDED BETWEEN THESE SIDES IS

$$A = \frac{1}{2}ab \text{ SIN } (\angle C)$$

Proof:-

LET Δ ABC BE GIVEN SUCH THAT BC = a AND AC = b.

Case i LET $\angle$C BE AN ACUTE ANGLE.

CONSIDER THE HEIGHT h DRAWN FROM B TO AC. IT MEETS AC AT D (SEE FIGURE 5.114).

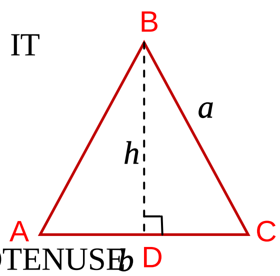


Figure 5.114

NOW, AREA OF ΔABC = $\frac{1}{2}bh$ ... (1)

SINCE Δ BCD IS RIGHT-ANGLED WITH HYPOTENUSE a,

$\text{SIN } (\angle C) = \frac{h}{a}$

$\therefore\ h = a \text{ SIN } (\angle C)$

REPLACING h BY a SIN ($\angle$C) IN 1 WE OBTAIN

AREA OF ΔABC = $\frac{1}{2}ab \text{ SIN}(\angle C)$ AS REQUIRED.

Case ii LET ∠C BE AN OBTUSE ANGLE.

DRAW THE HEIGHT FROM B TO THE EXTENDED BASE AC. IT MEETS THE EXTENDED AT D. NOW,

AREA OF △ABC = AREA OF △ABD – AREA OF △BDC

$$= \frac{1}{2}\text{AD}.h - \frac{1}{2}\text{CD}.h = \frac{1}{2}h(\text{AD} - \text{CD})$$

$$= \frac{1}{2}h.\text{AC} = \frac{1}{2}hb \qquad (2)$$

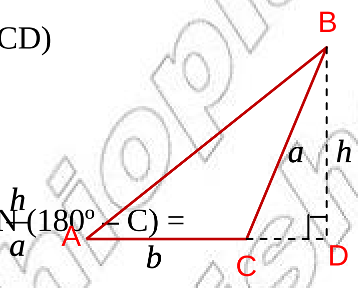


Figure 5.115

IN THE RIGHT-ANGLED TRIANGLE BCD, $\text{SIN}(180° - C) = \frac{h}{a}$

$$\therefore \quad h = a\ \text{SIN}\ (180° - C)$$

SINCE SIN (180° – C) = SIN C, WE HAVE $h = a$ SIN (∠C)

For any two angles A and B if m (∠A) + m (∠B) = 180°, then, sin A = sin B.

∴ REPLACING h BY a SIN (∠C) IN 2 WE OBTAIN;

AREA OF △ABC = $\frac{1}{2}ab$ SIN (∠C) AS REQUIRED.

Case iii LET ∠C BE A RIGHT ANGLE.

$$A = \frac{1}{2}ab = \frac{1}{2}ab(\text{SIN } 90°) \qquad (\text{SIN } 90° = 1)$$

$$= \frac{1}{2}ab\,\text{SIN}(\angle C) \qquad (\textit{as required})$$

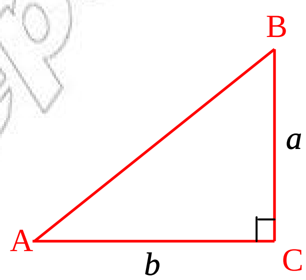


Figure 5.116

THIS COMPLETES THE PROOF.

Group Work 5.6

1 USING THIS FORMULA, SHOW THAT THE AREA A OF A REGULAR n-SIDED POLYGON WITH RADIUS r IS GIVEN BY

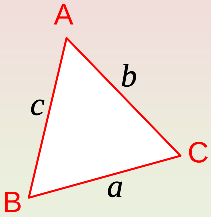
$$A = \frac{1}{2}nr^2\ \text{SIN}\frac{360^o}{n}$$

2 SHOW THAT THE AREA A OF AN EQUILATERAL TRIANGLE INSCRIBED IN A CIRCLE OF RADIUS r IS

$$A = \frac{3\sqrt{3}}{4}r^2$$

NOW WE STATE ANOTHER FORMULA CALLED **Heron's formula**, WHICH IS OFTEN USED TO FIND THE AREA OF A TRIANGLE WHEN ITS THREE SIDES ARE GIVEN.

Theorem 5.14 Heron's formula



THE AREA A OF A TRIANGLE WITH SIDES a, b AND c UNITS LONG AND SEMI- PERIMETER $s = \frac{1}{2}(a+b+c)$ IS GIVEN BY

$$A = \sqrt{s(s-a)(s-b)(s-c)}\ .$$

Figure 5.117

EXAMPLE 1 GIVEN ΔABC. IF AB = 15 UNITS, BC = 14 UNITS AND AC = 13 UNITS, FIND

A THE AREA OF ΔABC.

B THE LENGTH OF THE ALTITUDE FROM THE VERTEX A.

C THE MEASURE OF $\angle$B.

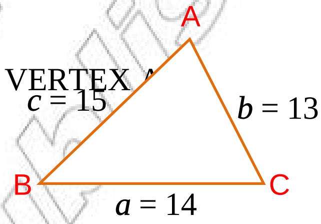


Figure 5.118

SOLUTION:

A $a = 14$ $\quad s - a = 7$

$b = 13$ $\quad s - b = 8$

$c = 15$ $\quad s - c = 6$

$a + b + c = 42$ $\quad (s - a) + (s - b) + (s - c) = 21$

Why is the sum of s – a, s – b, s – c equal to s? This provides a useful check.

$\therefore\ s = \dfrac{a+b+c}{2} = \dfrac{42}{2} = 21$

$\therefore$ AREA OF ΔABC = $\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(7)(8)(6)} = 84$ UNIT^2

B LET THE ALTITUDE FROM THE VERTEX A TO BC CORRESPONDING TO BASE BC BE h, MEETING BC AT D AS SHOWN.

THEN, AREA OF ΔABC = $\frac{1}{2}$ BC $\times h$

$\therefore\ 84 = \frac{1}{2} \times 14 \times h = 7h$

$\therefore\ h = \dfrac{84}{7} = 12$ UNITS.

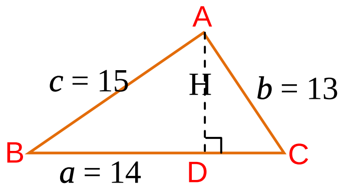


Figure 5.119

THEREFORE, THE ALTITUDE OF ΔABC FROM THE VERTEX A IS 12 UNITS LONG.

C IN THE RIGHT-ANGLED TRIANGLE ABD SHOWN ABOVE IN FIGURE 5.119, WE SEE THAT

$$\sin(\angle B) = \frac{AD}{AB} = \frac{h}{c} = \frac{12}{15} = 0.8$$

THEN, FROM TRIGONOMETRIC TABLES, WE FIND THAT THE CORRESPONDING ANGLE IS 53°, I.E., $m(\angle B) = 53^o$.

B Area of parallelograms

ACTIVITY 5.17

1 WHAT IS A PARALLELOGRAM?
2 SHOW THAT A DIAGONAL OF A PARALLELOGRAM DIVIDES THE PARALLELOGRAM INTO TWO CONGRUENT TRIANGLES.

Theorem 5.15

The area A of a parallelogram with base b and perpendicular height h is

$$A = bh$$

Proof:-

LET ABCD BE A PARALLELOGRAM WITH BASE BC = b. DRAW DIAGONAL AC. YOU KNOW THAT AC DIVIDES THE PARALLELOGRAM INTO TWO CONGRUENT TRIANGLES. MOREOVER, NOTE THAT ANY TWO CONGRUENT TRIANGLES HAVE EQUAL AREAS. NOW, THE AREA OF $\Delta ABC = \frac{1}{2}bh$.

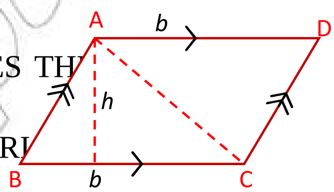


Figure 5.120

THEREFORE, THE AREA OF PARALLELOGRAM $= 2\left(\frac{1}{2}bh\right) = bh.$

EXAMPLE 2 IF ONE PAIR OF OPPOSITE SIDES OF A PARALLELOGRAM HAVE LENGTH 40 CM AND THE DISTANCE BETWEEN THEM IS 15 CM, FIND THE AREA OF THE PARALLELOGRAM.

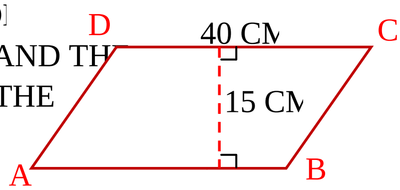


Figure 5.121

SOLUTION: AREA = 40 CM × 15 CM = 600 CM²

Exercise 5.13

1 IN ΔABC, $\overline{BE}$ AND $\overline{CF}$ ARE ALTITUDES OF THE TRIANGLE. IF AB = 6 UNITS, AC = 5 UNITS AND CF = 4 UNITS FIND THE LENGTH OF BE.

2 IN ΔDEF, IF DE = 20 UNITS, EF = 21 UNITS AND DF = 13 UNITS, FIND:

A THE AREA OF ΔDEF

B THE LENGTH OF ALTITUDE FROM THE VERTEX D

C SIN (∠D)

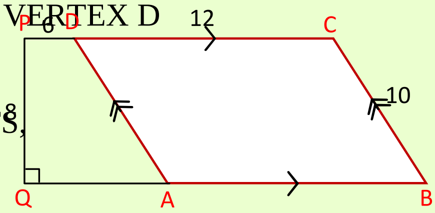


Figure 5.122

3 IN THE GIVEN FIGURE, PD = 6 UNITS, DC = 12 UNITS, PQ = 8 UNITS AND BC = 10 UNITS:

A THE AREA OF THE PARALLELOGRAM

B THE HEIGHT OF THE PARALLELOGRAM THAT CORRESPONDS TO AD.

4 PQRS IS A PARALLELOGRAM OF AREA 40 CM^2. IF PQ = 5 CM AND QR = 4 CM, CALCULATE THE LENGTHS OF THE CORRESPONDING ALTITUDES.

5 IN ΔMNO IF MN = 5 CM, NO = 6 CM AND MO = 7 CM, FIND:

A THE AREA OF ΔMNO. **B** THE LENGTH OF THE SHORTEST ALTITUDE.

(*leave your answers in radical form.*)

6 IN THE PARALLELOGRAM (SHOWN IN FIGURE 5.123 BELOW), AB = 2 CM, AD = 3 CM AND M(∠B) = 60°. FIND THE LENGTH OF THE ALTITUDE TO $\overline{DC}$.

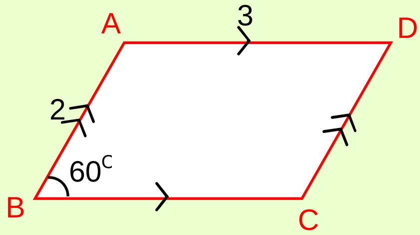


Figure 5.123

7 THE LENGTHS OF THREE SIDES OF A TRIANGLE ARE x, $4x$ AND $3x$ INCHES AND THE PERIMETER OF THE TRIANGLE IS 24 INCHES. FIND:

A THE LENGTHS OF THE SIDES OF THE TRIANGLE.

B THE AREA OF THE TRIANGLE.

8 FIND THE AREA OF A RHOMBUS WHOSE DIAGONALS ARE 5 INCHES AND 8 INCHES.

5.5.2 Further on Surface Areas and Volumes of Cylinders and Prisms

ACTIVITY 5.18

1 WHAT IS A SOLID FIGURE?

2 WHICH OF THE FOLLOWING SOLIDS ARE PRISMS AND WHICH ARE CYLINDERS? WHICH OF THEM ARE NEITHER PRISMS NOR CYLINDERS?

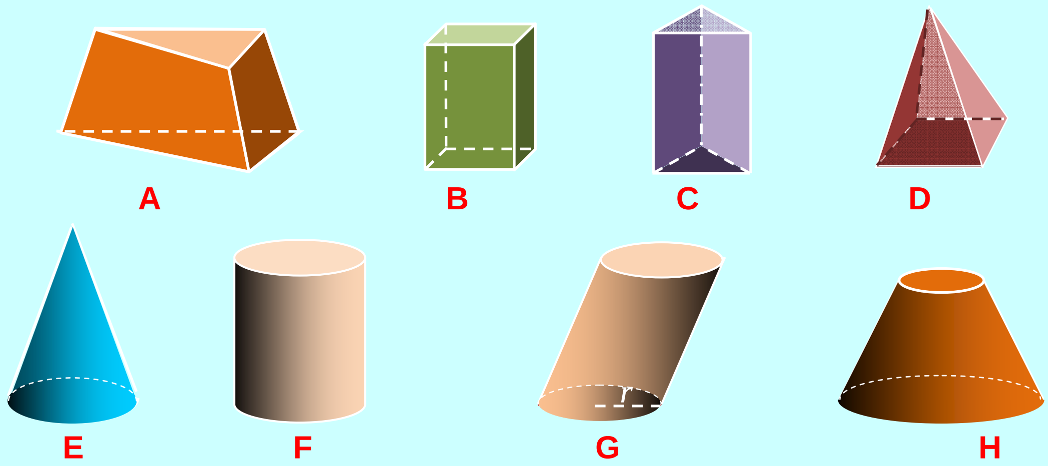


Figure 5.124

3 THE RADIUS OF THE BASE OF A RIGHT CIRCULAR CYLINDER IS 2 CM AND ITS ALTITUDE IS 3 CM. FIND ITS:

A CURVED SURFACE AREA B TOTAL SURFACE AREA C VOLUME

4 FIND A FORMULA FOR THE SURFACE AREA OF A RIGHT PRISM BY CONSTRUCTING A MODEL FROM SIMPLE MATERIALS.

5 ROLL A RECTANGULAR PIECE IN TO A CYLINDER. DISCUSS THE SURFACE AREA OF A RIGHT CIRCULAR CYLINDER.

A Prism

- A prism IS A SOLID FIGURE FORMED BY TWO CONGRUENT POLYGONAL REGIONS IN PARALLEL PLANES, ALONG WITH THREE OR MORE PARALLELOGRAMS, JOINING THE TWO PO… POLYGONS IN PARALLEL PLANES ARE CALLED bases.
- A PRISM IS NAMED BY ITS BASE. THUS, A PRISM IS CALLED TRIANGULAR, RECTANGULAR, PENTAGONAL, ETC., IF ITS BASE IS A TRIANGLE, A RECTANGLE, A PENTAGON, ETC., …
- IN A PRISM,
  - ✓ THE LATERAL EDGES ARE EQUAL AND PARALLEL.
  - ✓ THE LATERAL FACES ARE PARALLELOGRAMS.
- A right prism IS A PRISM IN WHICH THE BASE IS PERPENDICULAR TO THE LATERAL EDGE. OTHERWISE IT IS AN oblique prism.
- IN A RIGHT PRISM
  - ✓ ALL THE LATERAL EDGES ARE PERPENDICULAR TO BOTH BASES.
  - ✓ THE LATERAL FACES ARE RECTANGLES.
  - ✓ THE ALTITUDE IS EQUAL TO THE LENGTH OF EACH LATERAL EDGE.
- A REGULAR PRISM IS A RIGHT PRISM WHOSE BASE IS A REGULAR POLYGON.

Surface area and volume of prisms

- THE LATERAL SURFACE AREA OF A PRISM IS THE SUM OF THE AREAS OF ITS LATERAL FACES.
- THE TOTAL SURFACE AREA OF A PRISM IS THE SUM OF ITS LATERAL AREA AND THE AREA OF THE BASES.
- THE VOLUME OF ANY PRISM IS EQUAL TO THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE.

✓ IF WE DENOTE THE LATERAL SURFACE AREA OF A PRISM BY A_L, THE TOTAL SURFACE AREA BY A_T, THE AREA OF THE BASE BY A_B AND ITS VOLUME BY V, THEN

I $A_L = Ph$

WHERE P IS THE PERIMETER OF THE BASE AND h IS THE ALTITUDE OR HEIGHT OF THE PRISM.

II $A_T = 2A_B + A_L$ **III** $V = A_B\, h.$

EXAMPLE 1 THE ALTITUDE OF A RECTANGULAR PRISM IS 4 UNITS AND THE WIDTH AND LENGTH OF ITS BASE ARE 3 AND 2 UNITS RESPECTIVELY. FIND:

A THE TOTAL SURFACE AREA OF THE PRISM. **B** THE VOLUME OF THE PRISM.

SOLUTION:

A TO FIND A_T, FIRST WE HAVE TO FIND THE BASE AREA AND THE LATERAL AREA.

$A_B = 2 \times 3 = 6$ UNIT^2

AND $A_L = Ph = (3 + 2 + 3 + 2) \times 4 = 40$ UNIT^2

$\therefore A_T = 2A_B + A_L = 2 \times 6 + 40 = 52$ UNIT^2

SO, THE TOTAL SURFACE AREA IS 52 UNIT^2

B $V = A_B\, h = 6 \times 4 = 24$ UNIT^3

EXAMPLE 2 THROUGH THE CENTRE OF A REGULAR HEXAGONAL PRISM WHOSE BASE EDGE IS 6 CM AND HEIGHT 8 CM, A HOLE WHOSE FORM IS A REGULAR TRIANGULAR PRISM WITH BASE EDGE 3 CM IS DRILLED AS SHOWN IN FIGURE 5.125. FIND:

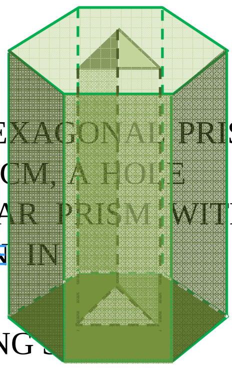


Figure 5.125

A THE TOTAL SURFACE AREA OF THE REMAINING SOLID.

B THE VOLUME OF THE REMAINING SOLID.

SOLUTION: RECALL THAT THE AREA OF A REGULAR n-SIDED POLYGON WITH RADIUS r IS

$$A = \frac{1}{2} nr^2 \sin\frac{360^o}{n}.$$

ALSO, THE RADIUS AND THE LENGTH OF A SIDE OF A REGULAR HEXAGON ARE EQU

SO, AREA OF THE GIVEN REGULAR HEXAGON $= \frac{1}{2} \times 6 \times 6^2 \times \sin 60^o = 54\sqrt{3}$ CM^2.

AREA OF THE EQUILATERAL TRIANGLE $= \frac{1}{2} ab \sin C = \frac{1}{2} \times 3 \times 3 \times \sin 60^o = \frac{9\sqrt{3}}{4}$ CM^2

A **i** AREA OF THE BASES OF THE REMAINING SOLID = 2 × (AREA OF HEXAGON – AREA OF Δ)

$$= 2 \times \left(54\sqrt{3} - \frac{9\sqrt{3}}{4}\right) = 108\sqrt{3} - \frac{9\sqrt{3}}{2} = \frac{207}{2}\sqrt{3}\ \text{CM}^2$$

ii LATERAL SURFACE AREA OF THE REMAINING SOLID = LATERAL AREA OF HEXAGONAL PRISM + LATERAL AREA OF TRIANGULAR PRISM (INNER)

= PERIMETER OF HEXAGON × 8 + PERIMETER OF TRIANGLE × 8

$= 36 \times 8 + 9 \times 8 = 360\ \text{CM}^2.$

∴ TOTAL SURFACE AREA OF THE REMAINING SOLID = $\left(\frac{207}{2}\sqrt{3} + 360\right)$ CM².

B VOLUME OF THE REMAINING SOLID

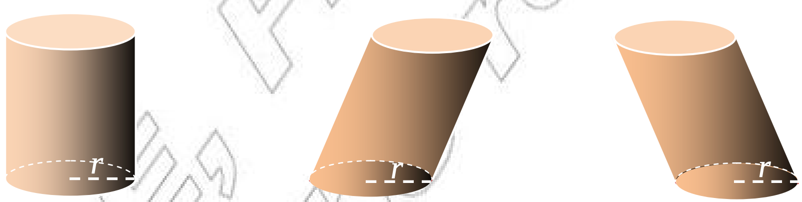
= VOLUME OF HEXAGONAL PRISM – VOLUME OF TRIANGULAR PRISM

$$= \left(54\sqrt{3} \times 8 - \frac{9\sqrt{3}}{4} \times 8\right) \text{CM}^3 = 414\sqrt{3}\ \text{CM}^3$$

B Cylinder

RECALL FROM YOUR LOWER GRADES THAT:

- A circular cylinder IS A SIMPLE CLOSED SURFACE BOUNDED ON TWO ENDS BY CIRCULAR BASES. (SEE FIGURE 5.126). A MORE GENERAL DEFINITION OF A CYLINDER REPLACES THE CIRCLE WITH ANY SIMPLE CLOSED CURVE. FOR EXAMPLE, THE CYLINDER SHOWN IN FIGURE 5.127 IS NOT A CIRCULAR CYLINDER.



A B C

Figure 5.126 (circular cylinders)

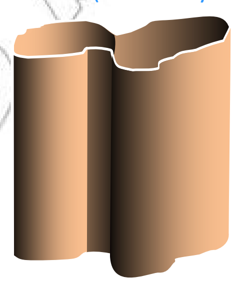


Figure 5.127

IN OUR PRESENT DISCUSSION, WE SHALL CONSIDER ONLY CYLINDERS WHOSE BASES ARE CIRCLES (I.E., CIRCULAR CYLINDERS).

A CIRCULAR CYLINDER RESEMBLES A PRISM EXCEPT THAT ITS BASES ARE CIRCUL FIGURE 5.126A THE CYLINDER IS CALLED A RIGHT CIRCULAR CYLINDER IF THE SUCH A CYLINDER IN THE LINE SEGMENT JOINING THE CENTRES OF THE BASES IS PERPENDICULAR TO THE CYLINDERS IN FIGURES 5.126B AND C ABOVE ARE NOT RIGHT CIRCULAR CYLINDERS; THEY A OBLIQUE CYLINDERS.

Surface area and volume of circular cylinders

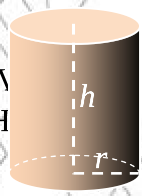


Figure 5.128

1 THE LATERAL SURFACE AREA (I.E., AREA OF THE CURVED SURFACE) OF A RIGHT CIRCULAR CYLINDER DENOTED BY A_L IS THE PRODUCT OF ITS HEIGHT h AND THE CIRCUMFERENCE C OF ITS BASE.

I.E. $A_L = hC$ OR $A_L = 2 \ rh$

2 THE TOTAL SURFACE AREA (OR SIMPLY SURFACE AREA) OF A RIGHT CIRCULAR CYLINDER DENOTED BY A_T IS TWO TIMES THE AREA OF THE CIRCULAR BASE PLUS THE AREA OF THE CURVED SURFACE (LATERAL SURFACE AREA). SO, IF THE HEIGHT OF THE CYLINDER IS h AND THE RADIUS OF THE BASE CIRCLE IS r, WE HAVE

$$A_T = 2 \ rh + 2 \ r^2 = 2 \ r(h + r)$$

3 THE VOLUME V OF THE RIGHT CIRCULAR CYLINDER IS EQUAL TO THE PRODUCT OF ITS BASE AREA AND HEIGHT.

SO, IF THE HEIGHT OF THE CYLINDER IS h AND ITS BASE RADIUS IS r, THEN

$$V = \ r^2 h$$

EXAMPLE 3 IF THE HEIGHT OF A RIGHT CIRCULAR CYLINDER IS 8 CM AND THE RADIUS OF ITS BASE IS 5 CM FIND THE FOLLOWING GIVING YOUR ANSWERS IN TERMS OF

A ITS LATERAL SURFACE AREA **B** ITS TOTAL SURFACE AREA **C** ITS VOLUME

SOLUTION:

A THE LATERAL SURFACE AREA OF THE RIGHT CIRCULAR CYLINDER IS GIVEN BY

$A_L = 2 \ rh$

$= 2 \ \times 5 \times 8 = 80 \ \text{CM}^2$

B $A_T = 2 \ rh + 2 \ r^2$

$= 2 \ \times 5 \times 8 + 2 \ \times 5^2 = 80 \ + 50 \ = 130 \ \text{CM}^2$

C THE VOLUME OF THE CYLINDER IS

$V = \ r^2 h$

$= \ \times 5^2 \times 8 = 200 \ \text{CM}^3$

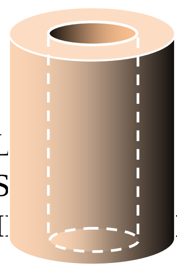


Figure 5.129

EXAMPLE 4 A CIRCULAR HOLE OF RADIUS 2 UNITS IS DRILLED THROUGH THE CENTRE OF A RIGHT CIRCULAR CYLINDER WHOSE BASE HAS RADIUS 3 UNITS AND WHOSE ALTITUDE IS 4 UNITS. FIND THE TOTAL SURFACE AREA OF THE RESULTING FIGURE.

SOLUTION: LET R BE THE RADIUS OF THE BIGGER CYLINDER AND r BE THE RADIUS OF THE SMALLER CYLINDER THEN

I AREA OF THE RESULTING BASE = 2 (R^2 – r^2)

$= 2 (\ \times 3^2 - \ \times 2^2)$ UNIT^2 = 10 UNIT^2

II LATERAL SURFACE AREA OF THE RESULTING FIGURE

= LATERAL SURFACE AREA OF THE BIGGER CYLINDER

+ LATERAL SURFACE AREA OF INNER (SMALLER) CYLINDER

= (2 Rh + 2 rh) UNIT^2 = [2 (3) 4 + 2 (2) 4] UNIT^2

= 40 UNIT^2

THEREFORE, TOTAL SURFACE AREA OF THE RESULTING FIGURE = (10 + 40) UNIT^2 = 50 UNIT^2

Exercise 5.14

1 USING THE MEASUREMENTS INDICATED IN EACH OF THE FIGURES, FIND:

A THE TOTAL SURFACE AREA OF EACH FIGURE. **B** THE VOLUME OF EACH FIGURE.

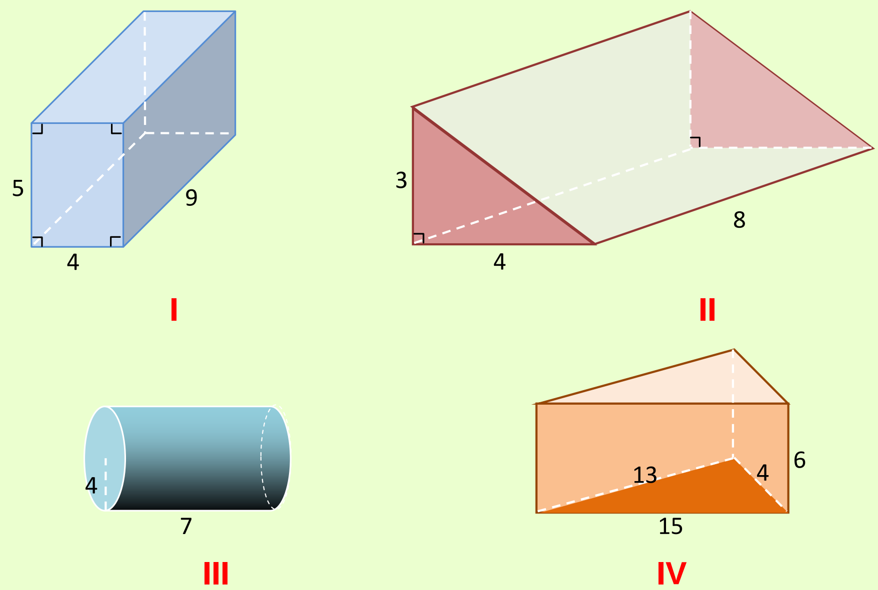


Figure 5.130

2 THE BASE OF A RIGHT PRISM IS AN ISOSCELES TRIANGLE WITH EQUAL SIDES 5 INCHES EACH, AND THIRD SIDE 4 INCHES. THE ALTITUDE OF THE PRISM IS 6 INCHES. FIND:

A THE TOTAL SURFACE AREA OF THE PRISM. **B** THE VOLUME OF THE PRISM.

3 FIND THE LATERAL SURFACE AREA AND TOTAL SURFACE AREA OF A RIGHT CIRCULAR CYLINDER IN WHICH:

A r = 4 FT, h = 12 FT **B** r = 6.5 CM, h = 10 CM

4 THROUGH A REGULAR HEXAGONAL PRISM WH 8 CM AND WHOSE HEIGHT IS CM, A HOLE IN THE SHAPE A RIGHT PRISM WITH ITS END BEING A RHOM DIAGONALS CM AND CM IS DRILLED (SEE FIGURE 5.131). FIND:

A THE TOTAL SURFACE AREA OF THE R

B THE VOLUME OF THE REMAI

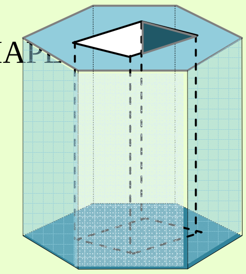


Figure 5.131

5 A MANUFACTURER MAKES A CLOSED RIGHT CYLINDRICAL CONTAIN 7 INCHES AND WHOSE HEIGHT M INCHES. HE ALSO MA ANOTHER CYLINDRICAL CONTAINER WHOSE BASE HA INCHES AND WHOSE HEIGHT MEASURES.

A WHICH CONTAINER REQUIRES MORE METAL?

B HOW MUCH MORE METAL DOES ? GIVE YOUR ANSWER IN T .

Key Terms

apothem	lateral surface area	sector
arc	parallelogram	segment
area	polygon	similarity
circle	prism	total surface area
congruency	regular polygon	triangle
cylinder	rhombus	volume

Summary

1 A POLYGON IS A SIMPLE CLOSED CURVE FORMED BY THE UNION OF SEGMENTS NO TWO OF WHICH IN SUCCESSION ARE COLLINEAR. T CALLED THE SIDES OF THE POLYGON AND THE END POINTS ITS vertices.

2 A A POLYGON IS SAID TO BE CONVEX IF EACH INTERIOR AN ^{o}.

B A POLYGON IS SAID TO BE C NON CONVEX), IF AT LEAST ONE OF ITS INTERIOR ANGLES IS GREATER ^{o}.

3 A DIAGONAL OF A POLYGON IS A LINE SEGMENT THAT OF ITS NON-CONSECUTIVE VE

4 A THE SUM OF ALL THE INTERIOR ANGLES OF AN n-SIDED POLYGON IS GIVEN I FORMULA

$$S = (n-2) \times 180^{o}$$

B THE SUM OF THE EXTERIOR ANGLES OF AN n-SIDED POLYGON IS GI

$$S = 360^{o}$$

5 A REGULAR POLYGON IS A CONVEX POLYGON WITH ALL SIDES EQUAL AND ALL ANGLES EQUAL.

6 **A** EACH INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS

$$\frac{(n-2)\times 180^o}{n}$$

B EACH EXTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$

C EACH CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$

7 A FIGURE HAS A LINE OF SYMMETRY, IF IT CAN BE FOLDED SO THAT ONE HALF OF THE FIGURE COINCIDES WITH THE OTHER HALF.

A FIGURE THAT HAS AT LEAST ONE LINE OF SYMMETRY IS CALLED A **symmetrical figure.**

8 AN n-SIDED REGULAR POLYGON HAS n LINES OF SYMMETRY.

9 A CIRCLE CAN BE ALWAYS INSCRIBED IN OR CIRCUMSCRIBED ABOUT ANY GIVEN REGULAR POLYGON.

10 THE APOTHEM IS THE DISTANCE FROM THE CENTRE OF A REGULAR POLYGON TO A SIDE OF THE POLYGON.

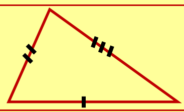
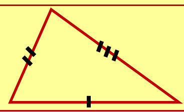
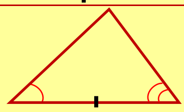
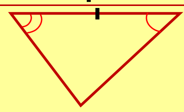
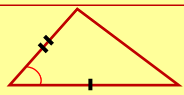
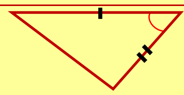
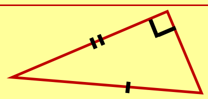
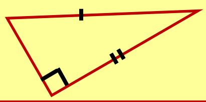
11 FORMULAE FOR THE LENGTH OF A SIDE s, APOTHEM a, PERIMETER P AND AREA A OF A REGULAR POLYGON WITH n SIDES AND RADIUS r ARE GIVEN BY

I $s = 2r \text{ SIN}\frac{180^o}{n}$ **II** $a = r \text{ COS}\frac{180^o}{n}$

III $P = 2nr \text{ SIN}\frac{180^o}{n}$ **IV** $A = \frac{1}{2}aP$ OR $A = \frac{1}{2}nr^2 \text{ SIN}\frac{360^o}{n}$

12 **Congruency**

TWO TRIANGLES ARE CONGRUENT, IF THE FOLLOWING CORRESPONDING PARTS OF ARE CONGRUENT.

I	THREE SIDES (SSS		
II	TWO ANGLES AN INCLUDED SIDE (ASA)		
III	TWO SIDES AND T INCLUDED ANGL (SAS)		
IV	A RIGHT ANGLE, HYPOTENUSE AN SIDE (RHS)		

13 Similarity

I TWO POLYGONS OF THE SAME NUMBER OF SIDES ARE SIMILAR, IF THEIR CORRESPONDING ANGLES ARE CONGRUENT AND THEIR CORRESPONDING SIDES HAVE THE SAME

II SIMILARITY OF TRIANGLES

A SSS-similarity theorem: IF THREE SIDES OF ONE TRIANGLE ARE PROPORTIONA TO THE THREE SIDES OF ANOTHER TRIANGLE, THEN THE TWO TRIANGLES

B SAS-similarity theorem: TWO TRIANGLES ARE SIMILAR, IF TWO PAIRS OF CORRESPONDING SIDES OF THE TRIANGLES ARE PROPORTIONAL AND TH ANGLES BETWEEN THE SIDES ARE CONGRUENT.

C AA-similarity theorem: IF TWO ANGLES OF ONE TRIANGLE ARE CORRESPONDINGLY CONGRUENT TO TWO ANGLES OF ANOTHER TRIANGLE TRIANGLES ARE SIMILAR.

14 IF THE RATIO OF THE LENGTHS OF ANY TWO CORRESPONDING SIDES OF TWO SIMILAR POLYGONS IS k THEN

I THE RATIO OF THEIR PERIMETERS IS k **II** THE RATIO OF THEIR AREAS IS k^2.

15 I Heron's formula

THE AREA A OF A TRIANGLE WITH SIDES a, b AND c UNITS LONG AND SEMI-PERIMETER

$s = \frac{1}{2}(a + b + c)$ IS GIVEN BY

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

II IF h IS THE HEIGHT OF THE TRIANGLE PERPENDICULAR TO b, THEN THE AREA OF THE TRIANGLE IS $A = \frac{1}{2} bh$

III IF THE ANGLE BETWEEN THE SIDES a AND b IS , THEN THE AREA OF THE TRIANGLE IS

$$A = \frac{1}{2} ab \text{ SIN}$$

16 RADIANS MEASURE ANGLES IN TERMS OF THE LENGTHS OF THE ARC SWEPT OUT BY THE ANGLE. A RADIAN (RAD) IS DEFINED AS THE MEASURE OF THE CENTRAL ANGLE SUBTEND OF A CIRCLE EQUAL TO THE RADIUS OF THE CIRCLE.

$$1 \text{ RADIAN} = \left(\frac{180}{\ }\right)^{O} \approx 57.3^{O}$$

$$1^{O} = \frac{\ }{180} \text{ RADIAN} \approx 0.0175 \text{ RADIAN.}$$

✓ TO CONVERT RADIANS TO DEGREE, MULTIPLY BY $\frac{180^{O}}{\ }$

✓ TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\ }{180^{O}}$

17 **I** FOR ANY ACUTE ANGLE

SIN = COS (90^o –)

COS = SIN (90^o –)

II FOR ANY ANGLE BETWEEN 90^o AND 180^o

SIN = SIN (180^o –)

COS = –COS (180^o –)

TAN = –TAN (180^o –)

18 **A** A CIRCLE IS SYMMETRICAL ABOUT EVERY DIAMETER.

B A DIAMETER PERPENDICULAR TO A CHORD BISECTS THE CHORD.

C THE PERPENDICULAR BISECTOR OF A CHORD PASSES THROUGH THE CENTRE OF THE CIRCLE.

D IN THE SAME CIRCLE, EQUAL CHORDS ARE EQUIDISTANT FROM THE CENTRE.

E A TANGENT IS PERPENDICULAR TO THE RADIUS DRAWN TO THE POINT OF CONTACT.

F LINE SEGMENTS THAT ARE TANGENTS TO A CIRCLE FROM AN OUTSIDE POINT ARE EQUAL.

19 **Angle properties of a circle**

A THE MEASURE OF AN ANGLE AT THE CENTRE IS TWICE THE MEASURE OF AN ANGLE AT THE CIRCUMFERENCE SUBTENDED BY THE SAME ARC.

B EVERY ANGLE AT THE CIRCUMFERENCE SUBTENDED BY A DIAMETER OF A CIRCLE IS A RIGHT ANGLE.

C INSCRIBED ANGLES IN THE SAME SEGMENT OF A CIRCLE ARE EQUAL.

20 **A** THE LENGTH ℓ OF AN ARC THAT SUBTENDS AN ANGLE AT THE CENTRE OF A CIRCLE WITH RADIUS r IS

$$\ell = \frac{R}{180^o}$$

B THE AREA A OF A SECTOR WITH CENTRAL ANGLE AND RADIUS r IS GIVEN BY

$$A = \frac{r^2}{360^o}$$

C THE AREA A OF A SEGMENT ASSOCIATED WITH CENTRAL ANGLE AND RADIUS R IS GIVEN BY

$$A = \frac{r^2}{360^0} - \frac{1}{2} r^2 \text{ SIN } \theta$$

21 IF A_L IS THE LATERAL SURFACE AREA OF THE PRISM, A_T IS THE TOTAL SURFACE AREA OF THE PRISM, A_B IS BASE AREA OF THE PRISM AND V IS THE VOLUME OF THE PRISM, THEN

I $A_L = Ph$, WHERE P IS THE PERIMETER OF THE BASE AND h IS THE ALTITUDE OR HEIGHT OF THE PRISM.

II $A_T = 2A_B + A_L$

III $V = A_B\, h$

Review Exercises on Unit 5

1. ABCDE is a pentagon. If m(∠A) = m(∠B) = m(∠C) = m(∠D) = 115°, find m(∠E).
2. Given a regular convex polygon with 20 sides, find the measure of:
   - **i** each interior angle. **ii** each exterior angle. **iii** each central angle.
3. The measure of each interior angle of a regular convex polygon is 150°. How many sides does it have?
4. The angles of a quadrilateral, taken in order, are $y°$, $3y°$, $5y°$, $7y°$. Verify that two of its sides are parallel.
5. Find the area of a regular hexagon if each side is 8 cm. (*Leave the answer in radical form*).
6. The area of a regular hexagon is given as $384\sqrt{3}$ cm^2.
   - **a** How long is each side of the hexagon?
   - **b** Find the radius of the hexagon. **c** Find the apothem of the hexagon.
7. Find the value of x in the following pair of congruent triangles:

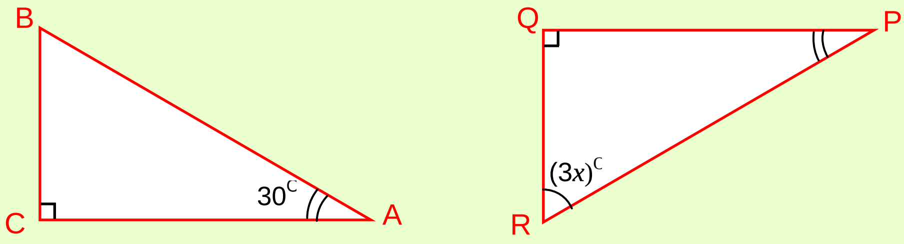


Figure 5.132

8. In Figure 5.133 below, BA = BC and KA = KC. Show that m(∠BAK) = m(∠BCK).

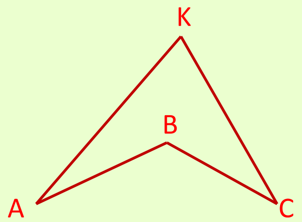


Figure 5.133

9. Two triangles are similar. The sides of one are 4, 6 and 7 cm. The shortest side of the other is 10 cm. Calculate the lengths of the other two sides of this
10. In the figure below, ∠ABC and ∠BDC are right angles; if AB = 5 cm, AD = 3 cm and BD = 4 cm, find BC and DC.

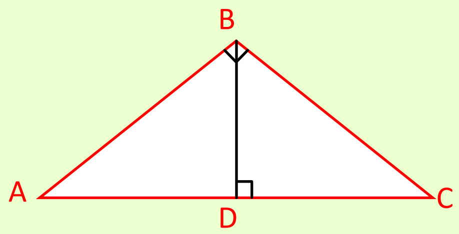


Figure 5.134

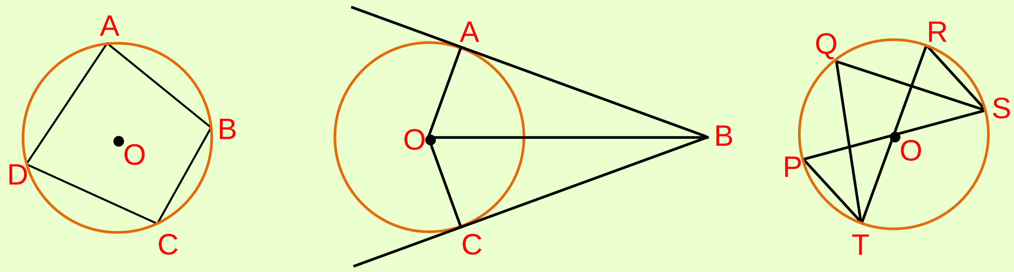
11 THE AREAS OF TWO SIMILAR TRIANGLES ARE 81 CM^2 AND 144 CM^2. IF ONE SIDE OF THE FIRST TRIANGLE IS 6 CM, WHAT IS THE LENGTH OF THE CORRESPONDING SIDE OF THE SEC

12 A CHORD OF A CIRCLE OF RADIUS 6 CM IS 8 CM LONG. FIND THE DISTANCE OF THE CHORD FROM THE CENTRE.

13 TWO CHORDS, AB AND CD, OF A CIRCLE INTERSECT AT RIGHT ANGLES AT A POINT INSIDE THE CIRCLE. IF m($\angle$BAC) = 35^o, FIND m($\angle$ABD).

14 IN EACH OF THE FOLLOWING FIGURES, O IS CENTRE OF THE CIRCLE. IN EACH FIGURE, IDENTIFY WHICH ANGLES ARE:

I SUPPLEMENTARY ANGLES. **II** RIGHT ANGLES. **III** CONGRUENT ANGLES.



A **B** **C**

Figure 5.135

15 FIND THE PERIMETER AND AREA OF A SEGMENT OF A CIRCLE OF RADIUS 8 CM, CUT OFF BY A CHORD THAT SUBTENDS A CENTRAL ANGLE OF:

A 120^o **B** $\frac{3}{4}$ RADIANS.

16 CALCULATE THE VOLUME AND TOTAL SURFACE AREA OF A RIGHT CIRCULAR CYLINDER OF HEIGHT 1 M AND RADIUS 70 CM.

17 A 40 M DEEP WELL WITH RADIUS $3\frac{1}{2}$ M IS DUG AND THE EARTH TAKEN OUT IS EVENLY SPREAD TO FORM A PLATFORM OF DIMENSIONS 28 M BY 22 M. FIND THE HEIGHT PLATFORM.

18 A GLASS CYLINDER WITH A RADIUS OF 7 CM HAS WATER UP TO A HEIGHT OF 9 CM. A METAL CUBE OF $5\frac{1}{2}$ CM EDGE IS IMMERSED IN IT COMPLETELY. CALCULATE THE HEIGHT BY THE WATER RISES IN THE CYLINDER.

19 AN AGRICULTURE FIELD IS RECTANGULAR, WITH DIMENSIONS 100 M BY 42 M. A 20 M DEEP WELL OF DIAMETER 14 M IS DUG IN A CORNER OF THE FIELD AND THE EARTH T SPREAD EVENLY OVER THE REMAINING PART OF THE FIELD. FIND THE INCREASE IN THE FIELD.