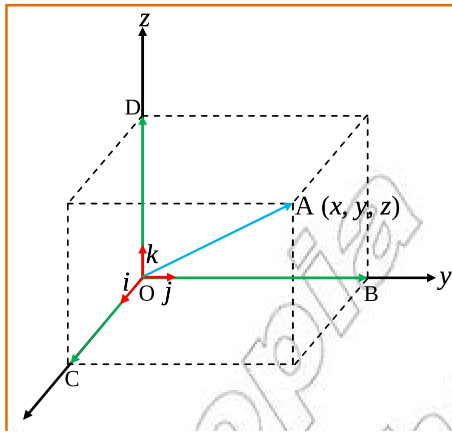
Unit 6



THREE DIMENSIONAL GEOMETRY AND VECTORS IN SPACE

Unit Outcomes:

After completing this unit, you should be able to:

- *know methods and procedures for setting up coordinate systems in space.*
- *know basic facts about coordinates and their use in determining geometric concepts in space.*
- *apply facts and principles about coordinates in space to solve related problems.*
- *know specific facts about vectors in space.*

Main Contents

INTRODUCTION

IN THIS UNIT, YOU WILL BE INTRODUCED TO THE COORDINATE SYSTEM IN SPACE WHICH IS AN OF THE COORDINATE SYSTEM ON THE PLANE THAT YOU ARE ALREADY FAMILIAR WITH. TH BEGINS WITH A SHORT REVISION OF THE COORDINATE PLANE AND THEN INTRODUCES T THREE DIMENSIONAL COORDINATE SYSTEM. YOU WILL LEARN HOW THE THREE DIMENSIONAL COORDINATES ARE USED TO FIND DISTANCE BETWEEN TWO POINTS, THE MIDPOINT OF A LINE SEGMENT IN SPACE HOW THEY ARE USED TO DERIVE THE EQUATION OF A SPHERE. FINALLY, YOU WILL SEE H DIMENSIONAL COORDINATES CAN BE APPLIED TO THE STUDY OF VECTORS IN SPACE.

EACH TOPIC IN THIS UNIT IS PRECEDED BY A FEW ACTIVITIES AND YOU ARE EXPECTED TO ATTE ACTIVITY. ATTEMPTING ALL THE EXERCISES AT THE END OF EACH SECTION WILL ALSO HELP Y WITH CONFIDENCE.

OPENING PROBLEM

TWO AIRPLANES TOOK OFF FROM THE SAME AIRPORT AT THE SAME TIME. ONE WAS HEADI WITH A GROUND SPEED OF 800KM/HR AND THE SECOND HEADING EAST WITH A GROUND SPEED OF 700KM/HR. IF THE FLIGHT LEVEL OF THE ONE HEADING NORTH IS 10KM AND THAT OF HEADI IS 12KM, WHAT IS THE DIRECT DISTANCE BETWEEN THE TWO AIRPLANES EXACTLY ONE HO TAKEOFF?

6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE

RECALL THAT YOU SET UP A RECTANGULAR COORDINATE SYSTEM ON A PLANE BY USING TWO THAT ARE PERPENDICULAR TO EACH OTHER AT A POINT O. ONE OF THEM, CALLED THE x-AXIS, IS MADE HORIZONTAL AND THE SECOND LINE, y-AXIS, IS MADE VERTICAL. THEN USING THESE TWO AXES YOU ASSOCIATE EACH POINT P OF THE PLANE WITH A UNIQUE ORDERED PAIR OF REAL NUMB AS (x, y).

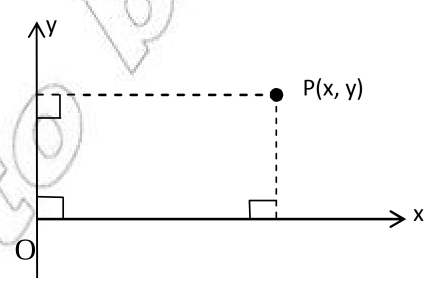


Figure 6.1

ACTIVITY 6.1

1 PLOT EACH OF THE FOLLOWING POINTS ON THE xy-COORDINATE PLANE.

A $P(2, 3)$ **B** $Q(-3, 3)$ **C** $R(0, -4)$

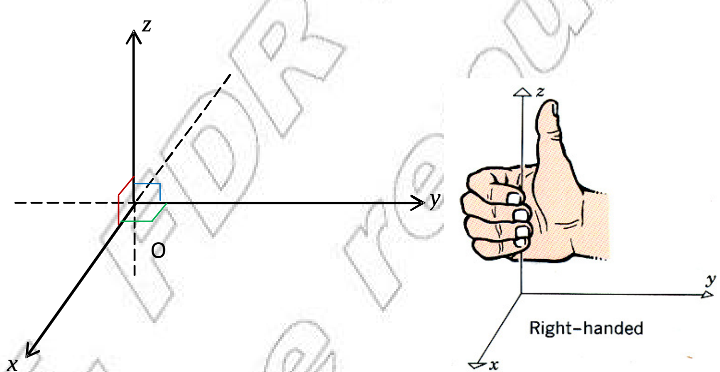
D $S(-2, -3)$ **E** $T\left(-\frac{3}{4}, \frac{1}{2}\right)$ **F** $U\left(\frac{3}{2}, -\frac{5}{2}\right)$

2 BY NAMING THE VERTICAL AXIS ON THE PLANE z, AND THE HORIZONTAL AXIS y, PLOT THE FOLLOWING POINTS.

A $A(2, 4)$ **B** $B(-2, 3)$ **C** $C(-3, -4)$

THIS ASSOCIATION OF THE POINTS OF THE PLANE AND ORDERED PAIRS OF REAL NUMBERS IS ONE TO ONE CORRESPONDENCE.

THE RECTANGULAR COORDINATE SYSTEM IS EXTENDED TO THREE DIMENSIONAL SPACES AS FOLLOWS. CONSIDER A FIXED POINT O IN SPACE AND THREE LINES THAT ARE MUTUALLY PERPENDICULAR AT THE POINT O. THE POINT O IS CALLED THE ORIGIN; THE THREE LINES ARE CALLED THE x-AXIS, THE y-AXIS AND THE z-AXIS. IT IS COMMON TO HAVE THE x- AND THE y-AXES ON A HORIZONTAL PLANE AND THE z-AXIS VERTICAL OR PERPENDICULAR TO THE PLANE CONTAINING THE x- AND y-AXES AT THE POINT O AS SHOWN IN FIGURE 6.2A BELOW. THE DIRECTIONS OF THE AXES ARE BASED ON THE RIGHT HAND RULE SHOWN IN FIGURE 6.2B BELOW.



A B

Figure 6.2

THE PLANE DETERMINED BY THE x- AND THE y-AXES IS CALLED THE xy-PLANE, THE PLANE DETERMINED BY THE x- AND THE z-AXES IS CALLED THE xz-PLANE AND THE PLANE DETERMINED BY THE y- AND THE z-AXES IS CALLED THE yz-PLANE. THESE THREE coordinate planes, WHICH INTERSECT AT THE ORIGIN, MAY BE VISUALIZED AS THE FLOOR OF A ROOM AND TWO ADJACENT WALLS OF THAT ROOM, WITH THE FLOOR REPRESENTING THE xy-PLANE, THE TWO WALLS CORRESPONDING TO xz- AND yz-PLANES THAT INTERSECT ON THE z-AXIS AND THE CORNER OF THE ROOM CORRESPONDING TO THE ORIGIN.

COMMONLY, THE POSITIVE DIRECTION OF THE x-AXIS IS COMING OUT OF THE PAGE TOWARDS THE READER; THE POSITIVE y-DIRECTION IS TO THE RIGHT AND THE POSITIVE z-DIRECTION IS UPWARDS. (OPPOSITE DIRECTIONS TO THESE ARE NEGATIVE).

NOTICE THAT THE coordinate planes PARTITION THE SPACE INTO EIGHT PARTS KNOWN AS octants. OCTANT 1 IS THE PART OF THE SPACE WHOSE BOUNDING EDGES ARE THE THREE POSITIV NAMELY, THE POSITIVE x-AXIS, THE POSITIVE y-AXIS AND THE POSITIVE z-AXIS. THEN OCTANTS 2, 3 AND 4 ARE THOSE WHICH LIE ABOVE THE xy-PLANE IN THE COUNTER CLOCKWISE ORDER ABOUT THE z-AXIS. OCTANTS 5, 6, 7 AND 8 ARE THOSE WHICH LIE BELOW THE xy-PLANE, WHERE OCTANT 5 IS JUST BELOW OCTANT 1 AND THE REST BEING IN THE COUNTER CLOCKWISE ORDER ABOUT THE z-AXIS AGAIN.

6.2 COORDINATES OF A POINT IN SPACE

AS INDICATED AT THE BEGINNING OF THIS UNIT, A POINT P ON A PLANE IS ASSOCIATED W UNIQUE ORDERED PAIR OF REAL NUMBERS (x, y) USING TWO PERPENDICULAR LINES KNOWN AS THE x-AXIS AND THE y-AXIS. YOU ALSO REMEMBER THAT x REPRESENTS THE DIRECTED DISTANCE OF P FROM THE y-AXIS AND y REPRESENTS THE DIRECTED DISTANCE OF P FROM THE x-AXIS. FOR EXAMPLE, IF THE COORDINATES OF A POINT P ARE $(-3, 2)$, IT MEANS THAT P IS 3 UNITS left OF THE y-AXIS AND 2 UNITS above THE x-AXIS. SIMILARLY, A POINT Q$(4, -5)$ IS FOUND 4 UNITS TO THE right OF THE y-AXIS AND 5 UNITS below THE x-AXIS.

ACTIVITY 6.2

PLOT EACH OF THE FOLLOWING POINTS USING THE THREE AXES INT ABOVE.

A A(3,4,0) **B** B(0,3,4) **C** C(3,0,4)

NOW A POINT P IN SPACE IS LOCATED BY SPECIFYING ITS DIRECTED DISTANCES FROM THE COORDINATE PLANES. ITS DIRECTED DISTANCE FROM THE yz-PLANE MEASURED ALONG OR PARALLEL TO THE x-AXIS IS ITS x-COORDINATE. ITS DIRECTED DISTANCE FROM THE xz-PLANE MEASURED ALONG OR IN THE DIRECTION OF THE y-AXIS IS ITS y-COORDINATE AND ITS DIRECTED DISTANCE MEASURED ALONG OR IN THE DIRECTION OF THE z-AXIS FROM THE xy-PLANE IS ITS z-COORDINATE.

THE COORDINATES OF P ARE THEREFORE WRITTEN AS an ordered triple (x, y, z) AS SHOWN IN FIGURE 6.3 BELOW.

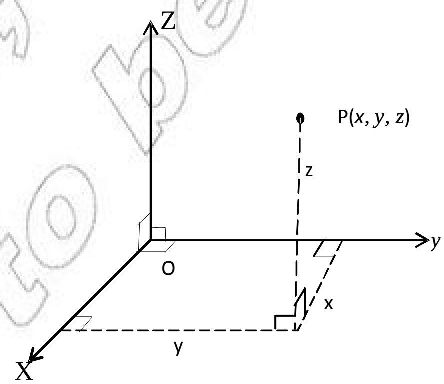


Figure 6.3

Example 1 LOCATE THE POINT A(2, 4, 3) IN SPACE USING THE REFERENCE AXES x, y AND z.

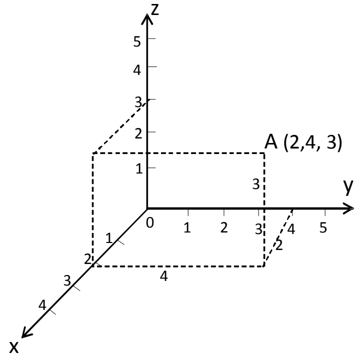


Figure 6.4 (*Example 1*)

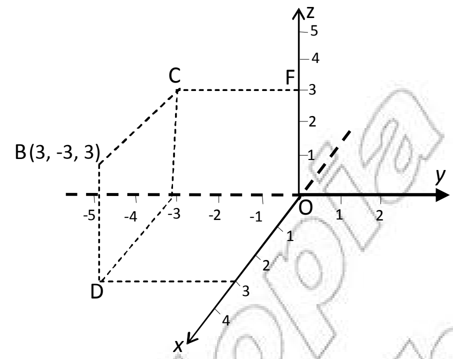


Figure 6.5 (*Example 2*)

Example 2 LOCATE THE POINT B(3, –3, 3) IN SPACE USING THE REFERENCE AXES x, y AND z.

THE PROCESS OF LOCATING THE POINT B MAY BE DESCRIBED AS FOLLOWS: START FRO ORIGIN O AND MOVE 3 UNITS IN THE DIRECTION OF THE POSITIVE x-AXIS. THEN MOVE 3 UNITS IN THE DIRECTION OF THE NEGATIVE y-AXIS AND FINALLY MOVE 3 UNITS UP IN THE DIRECTION OF THE POSITIVE z-AXIS TO GET POINT B.

ON THE SAME COORDINATE SYSTEM OF EXAMPLE 2 ABOVE, NOTICE THAT THE COORDINATES OF POINT C ARE (0, -3, 3), THE COORDINATES OF POINT D ARE (3, –3, 0), THE COORDINATES OF POINT F ARE (0, 0, 3) AND THE COORDINATES OF POINT O (OR THE ORIGIN) ARE (0, 0, 0).

LOCATING A GIVEN POINT IN SPACE AS OBSERVED FROM THE DIFFERENT EXAMPLES ABOVE CONSIDERED AS CORRESPONDING OR MATCHING A GIVEN ORDERED TRIPLE (x, y, z) OF REAL NUMBE WITH SOME POINT P IN SPACE. (SEE PROBLEMS 3, 4 *and* 5 *of* EXERCISE 6.1)

USING THIS FACT, IT IS POSSIBLE TO DESCRIBE SOME GEOMETRIC FIGURES IN SPACE BY ME EQUATIONS. FOR EXAMPLE, THE x-AXIS IS THE SET OF ALL POINTS IN SPACE WHOSE y AND z COORDINATES ARE ZERO. THUS WE EXPRESS IT AS FOLLOWS:

x-AXIS = $\{(x, y, z): x, y, z \in \mathbb{R}$ AND $y = z = 0\}$

Exercise 6.1

1 LOCATE EACH OF THE FOLLOWING POINTS IN SPACE USING THE REFERENCE AXES x, y AND z. YOU MAY USE THE SAME OR DIFFERENT COORDINATE SYSTEMS IN EACH CASE.

A P (3, 2, 3) **B** Q (–2, 4, 3) **C** R (3, – 3, 4)

D T(–2, –3,3) **E** M(0,0, –4) **F** N$\left(2.5, -\frac{1}{2}, -3\right)$

G Q(0,–3,0)

2 GIVE THE EQUATIONS OF

A THE y-AXIS **B** THE z-AXIS **C** THE xy-PLANE

D THE xz-PLANE **E** THE yz-PLANE

3 GIVEN ANY POINT P IN SPACE, DRAW A COORDINATE SYSTEM AND

- DROP A PERPENDICULAR LINE TO THE *xy*-PLANE.
- MARK THE INTERSECTION POINT OF THE PERPENDICULAR AND THE *xy*-PLANE BY Q.
- MEASURE THE DISTANCE FROM P TO Q WITH A RULER AND MARK THIS VALUE ON THE *z*- AXIS. CALL IT *z*.
- DROP PERPENDICULAR LINES TO THE *x*- AND *y*-AXES FROM POINT Q AND MARK THE INTERSECTION POINTS ON THE *x*- AND *y*-AXES, SAY *x* AND *y*.
- THE TRIPLE (*x*, *y*, *z*) YOU FOUND IN THE ABOVE STEPS UNIQUELY CORRESPONDS TO THE POINT. VERIFY! THUS (*x*, *y*, *z*) ARE THE COORDINATE OF P IN SPACE.

4 GIVEN ANY ORDERED TRIPLE (*a*, *b*, *c*):

- DRAW A COORDINATE SPACE AND LABEL EACH AXIS.
- MARK *a* ON THE *x*-AXIS, *b* ON THE *y*-AXIS AND *c* ON THE *z*-AXIS.
- FROM *a*, DRAW A LINE PARALLEL TO THE *y*-AXIS AND, FROM *b*, DRAW A LINE PARALLEL TO THE *x*-AXIS; FIND THE INTERSECTION OF THE TWO LINES: MARK IT AS POINT R.
- FROM POINT R, DRAW A LINE PARALLEL TO THE *z*-AXIS AND FROM *c*, DRAW A LINE PERPENDICULAR TO THE *z*-AXIS THAT INTERSECTS THE LINE FROM R. MARK THE INTERSECTION OF THESE TWO LINES BY POINT P.
- THE POINT P IN SPACE CORRESPONDS TO THE ORDERED TRIPLE (*a*, *b*, *c*). VERIFY THAT THERE IS NO OTHER POINT IN SPACE DESCRIBING THE ORDERED TRIPLE (*a*, *b*, *c*). THUS, (*a*, *b*, *c*) ARE THE COORDINATES OF P.

5 CAN YOU CONCLUDE FROM THE ABOVE TWO STATEMENTS, PROBLEMS 3 AND 4, THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF POINTS IN SPACE AND THE S ORDERED TRIPLES OF REAL NUMBERS? WHY? YOU MAY NEED TO USE THE BASIC FAC SOLID GEOMETRY ABOUT PARALLEL AND PERPENDICULAR LINES AND PLANES IN SPACE

6.3 DISTANCE BETWEEN TWO POINTS IN SPACE

OPENING PROBLEM

ASSUME THAT YOUR CLASSROOM IS A RECTANGULAR BOX WHERE THE FLOOR IS 8 METRES L METRES WIDE. IF THE DISTANCE FROM THE FLOOR TO THE CEILING (HEIGHT OF THE ROO METRES, FIND THE DIAGONAL DISTANCE BETWEEN A CORNER OF THE ROOM ON THE FLOO OPPOSITE CORNER ON THE CEILING.

AFTER COMPLETING THIS SECTION, YOU WILL SEE THAT SOLVING THIS PROBLEM IS A M FINDING DISTANCE BETWEEN TWO POINTS IN SPACE USING THEIR COORDINATES.

ACTIVITY 6.3

1 ON THE COORDINATE PLANE, CONSIDER POINTS P(x_1, y_1) AND Q(x_2, y_2) TO BE ANY TWO DISTINCT POINTS. THEN FIND THE DISTANCE BETWEEN P AND Q OR THE LENGTH OF THE LINE SEGMENT PQ BY USING THE PYTHAGORAS THEOREM.

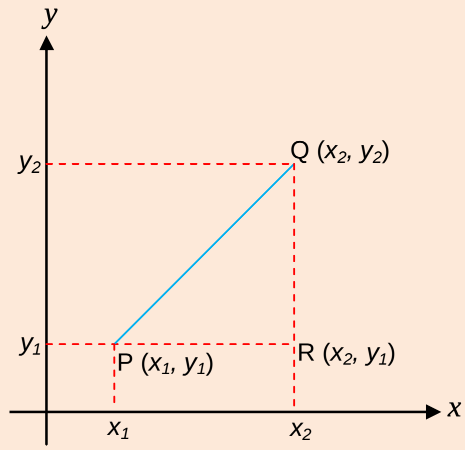


Figure 6.6

2 FIND THE DISTANCE BETWEEN THE FOLLOWING PAIRS OF POINTS.

A A(3,4,0) AND B(1,5,0) B C(0,3,4) AND D(0,1,2)

C E(4,0,5) AND F(1,0,1)

THE SAME PRINCIPLE WHICH YOU USE IN TWO DIMENSIONS CAN BE USED TO FIND THE DIS BETWEEN TWO POINTS IN SPACE WHOSE COORDINATES ARE GIVEN.

FIRST, LET US CONSIDER THE DISTANCE OF A POINT $P(x, y, z)$ FROM THE ORIGIN O OF THE COORDINATE SYSTEM.

FROM THE POINT $P(x, y, z)$, LET US DROP PERPENDICULAR LINE SEGMENTS TO THE THREE PLAN AND LET US COMPLETE THE RECTANGULAR BOX WHOSE EDGES ARE x, y AND z UNITS LONG AS SHOWN IN FIGURE 6.7. LET ITS VERTICES BE NAMED O, A,B,C, D, P, Q AND R.

THEN, TO FIND THE DISTANCE FROM O TO POINT P, CONSIDER THE RIGHT ANGLED TRIANGI AND OBP.

HERE NOTICE THAT $\overline{PB}$ IS PERPENDICULAR TO THE xy-PLANE AT B, AND HENCE IT IS PERPENDICULAR TO $\overline{OB}$ AT B.

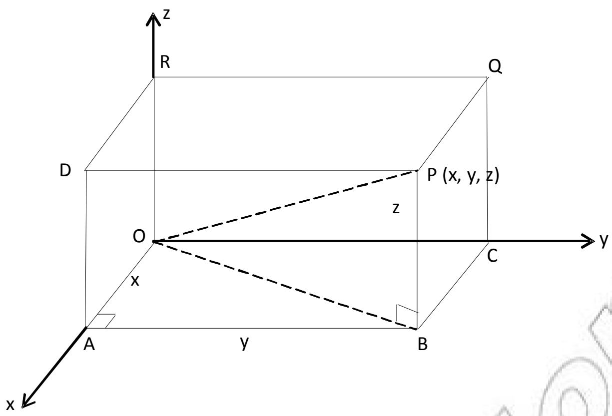


Figure 6.7

NOW AS $\overline{OP}$ IS THE HYPOTENUSE OF THE RIGHT ANGLED TRIANGLE OBP, YOU KNOW B
PYTHAGORAS THEOREM THAT $(OP)^2 = (OB)^2 + (PB)^2$

ONCE AGAIN, $\overline{OB}$ IS THE HYPOTENUSE OF THE RIGHT ANGLED TRIANGLE OAB, YOU HAVE

$$(OB)^2 = (OA)^2 + (AB)^2 .$$

THEN SUBSTITUTING $(OB)^2$ BY $(OA)^2 + (AB)^2$ IN $(OP)^2 = (OB)^2 + (PB)^2$, YOU OBTAIN

$$(OP)^2 = (OA)^2 + (AB)^2 + (PB)^2 = x^2 + y^2 + z^2$$

OR $\quad OP = \sqrt{x^2 + y^2 + z^2}$

Note:

OBSERVE THAT $\overline{OP}$ IS A DIAGONAL OF THE RECTANGULAR BOX AND x, y, z IN ABSOLUTE VALUE, ARE THE LENGTHS OF ITS THREE CONCURRENT EDGES. THEREFORE, THE DISTANCE FROM NOW THE LENGTH OF THE DIAGONAL OF THE RECTANGULAR BOX WHICH IS THE SQUARE I SUM OF THE SQUARES OF THE LENGTHS OF THE THREE EDGES OF THE BOX

Example 1 FIND THE DISTANCE FROM THE ORIGIN TO THE POINT P(3, 4, 5).

Solution THE DISTANCE FROM THE ORIGIN TO THE POINT P IS THE LENGTH OF THE LINE SEGMENT $\overline{OP}$, WHICH IS

$$OP = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2} \text{ UNITS}$$

Example 2 FIND THE DISTANCE FROM THE ORIGIN TO THE POINT $Q(-2, 0, 3)$

Solution $OQ = \sqrt{(-2)^2 + 0^2 + 3^2} = \sqrt{13}$ UNITS

NOW, LET $P(x_1, y_1, z_1)$ AND $Q(x_2, y_2, z_2)$ BE ANY TWO POINTS IN SPACE. TO FIND THE DISTANCE BETWEEN THESE TWO GIVEN POINTS, YOU MAY CONSIDER A RECTANGULAR BOX IN THE CO SPACE SO THAT THE GIVEN POINTS P AND Q ARE ITS OPPOSITE VERTICES, $\overline{PQ}$ IS THE DIAGONAL AS SHOWN IN FIGURE 6.8.

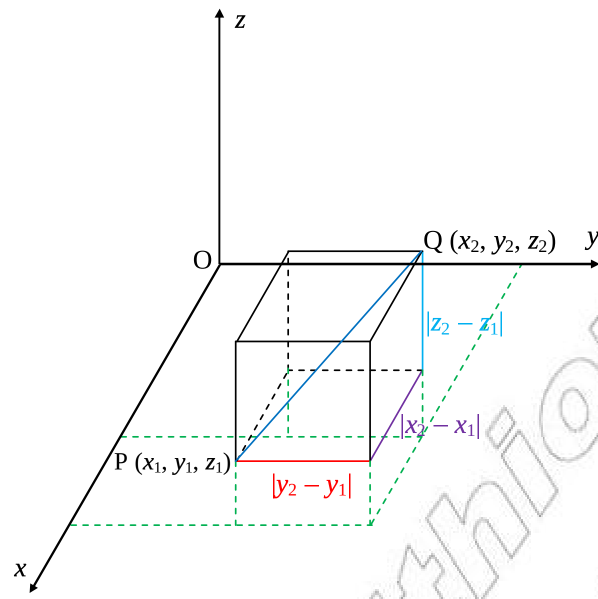


Figure 6.8

THEN WE SEE THAT THE LENGTHS OF THE THREE CONCURRENT EDGES OF THE BOX ARE $|x_2 - x_1|$, $|y_2 - y_1|$ AND $|z_2 - z_1|$.

THUS, THE DISTANCE FROM P TO Q OR THE LENGTH OF THE DIAGONAL $\overline{PQ}$ OF THE BOX, IS GIVEN BY

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3 FIND THE DISTANCE BETWEEN THE POINTS $P(1,-2,3)$ AND $Q(-4,0,5)$

Solution $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(-4-1)^2 + (0-(-2))^2 + (5-3)^2}$

$= \sqrt{25+4+4} = \sqrt{33}$ UNITS

Exercise 6.2

1 FIND THE DISTANCE BETWEEN THE GIVEN POINTS IN SPACE.

A A(0,1,0) AND B(2,0,3)

B C(2,1,3) AND D(4,6,10)

C E(–1,–3,6) AND F(4,0,–2)

D G(7,0,0) AND H(0,–4,2)

E $L\left(-1,-\frac{1}{2},-\frac{1}{4}\right)$ AND M(–4,0,–1)

F N(7,11,12) AND P(–6,–2,0)

G $Q\left(\sqrt{2},-\sqrt{2},1\right)$ AND R(0,0,–11)

2 CAN YOU NOW SOLVE THE OPENING PROBLEM? PLEASE TRY IT.

6.4 MIDPOINT OF A LINE SEGMENT IN SPACE

ACTIVITY 6.4

ON THE COORDINATE PLANE, IF $P(x_1, y_1)$ AND $Q(x_2, y_2)$ ARE THE ENDPOINTS OF A LINE SEGMENT $\overline{PQ}$, YOU KNOW THAT ITS MIDPOINT M HAS COORDINATES $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

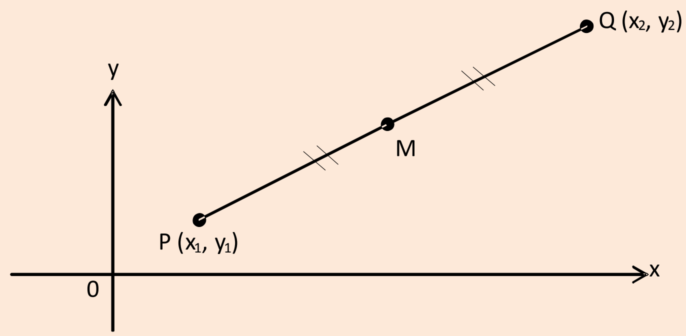


Figure 6.9

1 FIND THE COORDINATES OF THE MIDPOINTS OF THE LINE SEGMENTS WITH GIVEN END PO ON A PLANE.

A A(2,4) AND B(0,2) **B** C(–1,3) AND D(3,–1)

C E $\left(\frac{1}{4}, -\frac{3}{4}\right)$ AND F $\left(\frac{3}{4}, \frac{3}{4}\right)$

2 FIND THE COORDINATES OF THE MIDPOINTS OF THE LINE SEGMENTS WITH GIVEN END PO IN SPACE.

A A(2,4,0) AND B(0,2,0) **B** C (–1,3,0) AND D(3,–1,0)

C E $\left(\frac{1}{4}, -\frac{3}{4}, 0\right)$ AND F $\left(\frac{3}{4}, \frac{3}{4}, 0\right)$

THE COORDINATES OF THE MIDPOINT OF A LINE SEGMENT IN SPACE ARE ALSO OBTAINED IN T WAY. THAT IS, THE COORDINATES OF THE MIDPOINT ARE OBTAINED BY TAKING THE AVERA RESPECTIVE COORDINATES OF THE ENDPOINTS OF THE GIVEN LINE SEGMENT. THUS, IF P(x_1, y_1, z_1) AND Q(x_2, y_2, z_2) ARE THE END POINTS OF A LINE SEGMENT IN SPACE, THE COORDINATES OF MIDPOINT M WILL BE $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ *See* FIGURE 6.10.

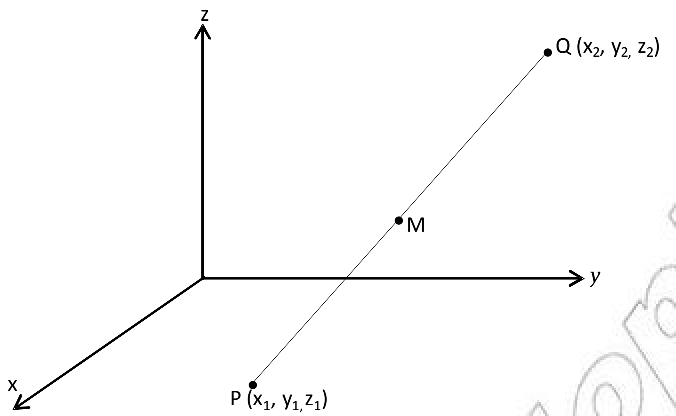


Figure 6.10

Example 1 FIND THE MIDPOINT OF THE LINE SEGMENT WITH ENDPOINTS A(0, 0, 0) AND B(4, 6, 2).

Solution THE MIDPOINT OF $\overline{AB}$ WILL BE AT THE POINT M WHOSE COORDINATES ARE

$$\left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{0+2}{2}\right) = (2, 3, 1).$$

THAT IS, M(2, 3, 1) IS THE MIDPOINT OF $\overline{AB}$.

Example 2 FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE ENDPOINTS ARE P(–1, 3, –3) AND Q(1, 5, 7).

Solution THE MIDPOINT OF $\overline{PQ}$ IS AT THE POINT M WHOSE COORDINATES ARE

$$\left(\frac{-1+1}{2}, \frac{3+5}{2}, \frac{-3+7}{2}\right) = (0, 4, 2).$$

SO, THE POINT M(0, 4, 2) IS THE MIDPOINT OF $\overline{PQ}$.

Exercise 6.3

1 FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE ENDPOINTS ARE:

- **A** A (1, 3, 5) AND B (3, 1, 1)
- **B** P (0, –2, 2) AND Q (–4, 2, 4)
- **C** $C\left(\frac{1}{2}, 3, 0\right)$ AND $D\left(\frac{3}{4}, -1, 1\right)$
- **D** R (0, 9, 0) AND S (0, 0, 8)
- **E** T (–2, –3, –5) AND U (–1, –1, –7)
- **F** G (6, 0, 0) AND H (0, –4, –2)
- **G** $M\left(\frac{1}{2}, \frac{1}{3}, -1\right)$ AND $K\left(-\frac{1}{2}, 0, \frac{1}{4}\right)$

2 IF THE MIDPOINT OF A LINE SEGMENT IS AT M(2, 5, –3) AND ONE OF ITS ENDPOINTS IS AT R (–3, 2, 4), FIND THE COORDINATES OF THE OTHER ENDPOINT.

6.5 EQUATION OF SPHERE

ACTIVITY 6.5

WHEN THE CENTRE OF A CIRCLE IS AT (*h*, *k*) AND ITS RADIUS IS *r*, THE EQUATION OF THE CIRCLE IS GIVEN BY $(x-h)^2 + (y-k)^2 = r^2$

HERE NOTICE THAT C(*h*, *k*) IS THE CENTRE AND P(*x*, *y*) IS ANY POINT ON THE CIRCLE AND *r* IS THE RADIUS OF THE CIRCLE OR THE DISTANCE OF P(*x*, *y*) FROM THE CENTRE C(*h*, *k*).

NOW USING SIMILAR NOTIONS:

1 DEFINE A SPHERE WHOSE RADIUS IS *r* AND WHOSE CENTRE IS AT (*a*, *b*, *c*).

2 A FIND THE EQUATION OF THE SPHERE WHOSE CENTRE IS AT THE ORIGIN, AND HAS RA $r = 2$.

B IF A POINT P(*x*, *y*, *z*) IS ON THE SURFACE OF THIS SPHERE, WHAT IS THE DISTANCE OF P FROM THE CENTRE OF THE SPHERE?

3 IF THE CENTRE OF A SPHERE IS AT THE ORIGIN AND ITS RADIUS IS *r*, WHAT IS THE DISTANCE OF A POINT P(3,4,0) ON THE SURFACE OF THE SPHERE FROM THE ORIGIN?

NOW, LET US CONSIDER A SPHERE WHOSE CENTRE IS AT THE ORIGIN OF A COORDINATE SY WHOSE RADIUS IS *r*. THEN, IF P(*x*, *y*, *z*) IS ANY POINT ON THE SURFACE OF THE SPHERE, THE LENGTH OF $\overline{OP}$ IS THE RADIUS OF THAT SPHERE. IN THE DISCUSSION ABOVE, YOU HAVE SEEN THA LENGTH OF $\overline{OP}$ IS GIVEN BY $\sqrt{x^2 + y^2 + z^2}$. THEREFORE, EVERY POINT P(*x*, *y*, *z*) ON THE SPHERE SATISFIES THE EQUATION $r = \sqrt{x^2 + y^2 + z^2}$.

THAT MEANS, IF THE CENTRE OF A SPHERE IS AT THE ORIGIN OF THE COORDINATE SPACE A *r* IS ITS RADIUS, THE EQUATION OF SUCH A SPHERE IS GIVEN BY $x^2 + y^2 + z^2 = r^2$

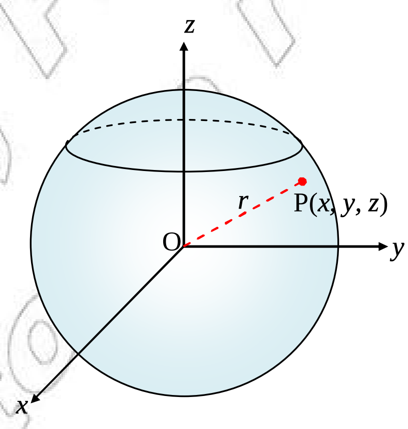


Figure 6.11

Example 1 WRITE THE EQUATION OF THE SPHERE WHOSE CENTRE IS AT THE ORIGIN AND WH RADIUS IS 3 UNITS.

Solution IF P(x, y, z) IS ANY POINT ON THE SPHERE, ITS DISTANCE FROM THE ORIGIN (THE CENTRE) IS GIVEN BY $d = \sqrt{x^2 + y^2 + z^2}$. SUBSTITUTING d BY 3, WE GET THE EQUATION OF THE SPHERE TO BE $\sqrt{x^2 + y^2 + z^2} = 3$, WHICH IS EQUIVALENT TO $x^2 + y^2 + z^2 = 9$.

THEREFORE, THE EQUATION OF THE SPHERE WILL BE $x^2 + y^2 + z^2 = 9$

NOW LET US CONSIDER A SPHERE WHOSE CENTRE IS NOT AT THE ORIGIN BUT AT ANY OTHER C(a, b, c) IN SPACE. IF P(x, y, z) IS ANY POINT ON THE SURFACE OF THE SPHERE, THEN THE RADIUS OF THE SPHERE WILL BE THE LENGTH OF $\overline{CP}$.

THAT MEANS, IN THIS CASE $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

THEREFORE, THE EQUATION OF THE SPHERE IN THIS CASE IS

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

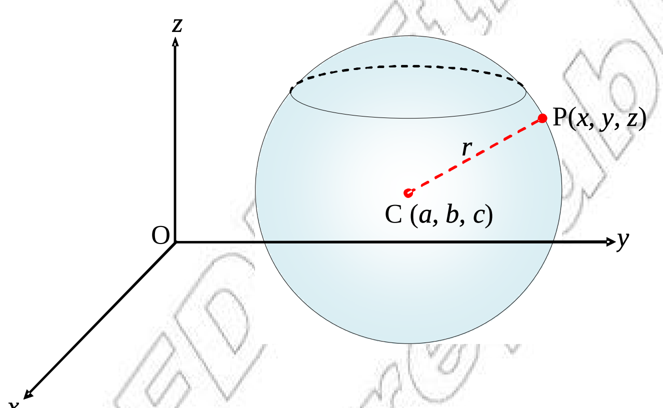


Figure 6.12

Example 2 WRITE THE EQUATION OF THE SPHERE WITH CENTRE AT C(1, 2, 3) AND RADIUS 4 UNITS.

Solution IF P(x, y, z) IS ANY POINT ON THE SURFACE OF THE SPHERE, THEN THE DISTANCE FROM THE CENTRE C TO THE POINT P IS GIVEN TO BE THE RADIUS OF THE SPHERE. THAT MEANS $r = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$

SUBSTITUTING r BY 4 AND SQUARING BOTH SIDES, YOU GET THE EQUATION OF THE SPHERE TO

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 16.$$

OBSERVE THAT WHEN THE CENTRE IS AT THE ORIGIN (0, 0, 0) THE EQUATION

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ REDUCES TO THE FORM $x^2 + y^2 + z^2 = r^2$.

(SUBSTITUTING (a, b, c) BY (0, 0, 0)).

THAT MEANS THE EQUATION OF A SPHERE GIVEN BY $(x-a)^2+(y-b)^2+(z-c)^2=r^2$, WHERE r IS THE RADIUS AND C(a, b, c) IS THE CENTRE CAN BE APPLIED TO A SPHERE WHOSE CENTRE IS AT ANY POINT C(a, b, c) INCLUDING THE ORIGIN.

Example 3 GIVEN THE EQUATION OF A SPHERE TO BE $x^2 + y^2 + z^2 = 9$, WHAT CAN YOU SAY ABOUT THE POINTS:

A P(1, 2, 2,)? **B** Q(0, 1, 2,)? **C** R(1, 3, 2)?

Solution CLEARLY THE CENTRE OF THE SPHERE IS O(0, 0, 0) AND ITS RADIUS IS 3.

A BECAUSE THE DISTANCE OF P FROM THE CENTRE IS 3, P IS ON THE SURFACE OF THE SPHERE.

B BECAUSE THE DISTANCE OF Q FROM THE CENTRE IS $\sqrt{5}$, WHICH IS LESS THAN 3, Q IS INSIDE THE SPHERE.

C BECAUSE THE DISTANCE OF R FROM THE CENTRE IS $\sqrt{14} > 3$, R IS OUTSIDE THE SPHERE.

In general, if O is the centre of a sphere and *r* is its radius,then for any point P taken in space, we have one of the following three possibilities.

I OP = r, IN WHICH CASE P IS ON THE SURFACE OF THE SPHERE;

II OP < r, IN WHICH CASE P IS INSIDE THE SPHERE; AND

III OP > r, IN WHICH CASE P IS OUTSIDE THE SPHERE.

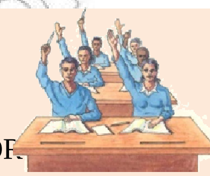
Exercise 6.4

1. WRITE THE EQUATION OF A SPHERE OF RADIUS 4 UNITS WHOSE CENTRE IS AT C(3, 0, 5).
2. GIVEN THE EQUATION OF A SPHERE TO BE $x^2 + y^2 + z^2 - 6x - 4y - 10z = -22$, FIND THE CENTRE AND RADIUS OF THE SPHERE.
3. IF A(0, 0, 0) AND B(4, 6, 0) ARE END POINTS OF A DIAMETER OF A SPHERE, WRITE ITS EQUATION.
4. HOW FAR IS THE POINT (–1, 2, 3) FROM THE SPHERE WHOSE EQUATION IS
   $(x-1)^2 + (y+2)^2 + z^2 = 1$?
5. IF THE CENTRE OF A SPHERE IS AT THE ORIGIN AND ITS RADIUS IS 10 UNITS, DETERMINE WHICH OF THE FOLLOWING POINTS LIE INSIDE OR OUTSIDE OR ON THE SPHERE.
   A(2, 1, 2) B(–3, 2, 4) C(5, 8, 6) D(0, 8, 6) E(–8, –6, 0)
6. DECIDE WHETHER OR NOT EACH OF THE FOLLOWING POINTS IS INSIDE, OUTSIDE OR ON THE SPHERE WHOSE EQUATION IS
   $x^2 + y^2 + z^2 + 2x - y + z = 0.$
   A O(0, 0, 0) **B** P(–1, 0, 1) **C** Q(0, $\frac{1}{2}$, 0)
7. **A** STATE THE COORDINATES OF ANY POINT IN SPACE WHICH IS ON THE x-AXIS?
   B FIND THE COORDINATES OF TWO POINTS ON THE x-AXIS WHICH ARE $\sqrt{2}$ UNITS FROM THE POINT P(–1, –1, 2).

6.6 VECTORS IN SPACE

RECALL THAT A VECTOR QUANTITY IS A QUANTITY THAT HAS BOTH MAGNITUDE AND DIRECTI
VELOCITY AND FORCE ARE EXAMPLES OF VECTOR QUANTITIES. ON THE OTHER HAND, A QUAN
MAGNITUDE ONLY BUT NO DIRECTION IS CALLED A SCALAR QUANTITY. FOR EXAMPLE, MASS AN
EXAMPLES OF SCALAR QUANTITIES.

ACTIVITY 6.6



1. HOW DO YOU REPRESENT A VECTOR ON A PLANE?
2. HOW DO YOU REPRESENT THE MAGNITUDE OF A VECTOR?
3. HOW DO YOU SHOW THE DIRECTION OF A VECTOR?
4. HOW DO YOU EXPRESS THE VECTOR IN FIGURE 6.13 BELOW USING THE STANDARD UNIT VECTORS **i** AND **j**?

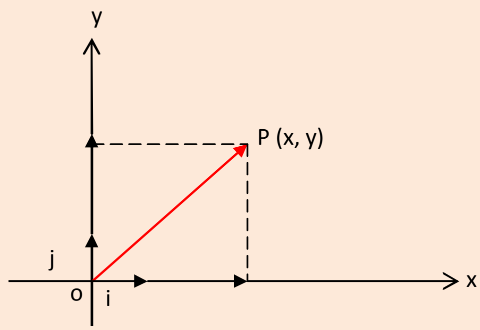


Figure 6.13

RECALL ALSO THAT THE VECTOR $\overrightarrow{OP}$ CAN BE NAMED USING A SINGLE LETTER. THAT IS, $\overrightarrow{OP}$ MAY BE NAMED $\vec{a}$, OR SIMPLY $\boldsymbol{a}$ SO THAT $\vec{a} = \boldsymbol{a} = x\mathbf{i} + y\mathbf{j}$

OPERATIONS ON VECTORS CAN BE PERFORMED USING THEIR COMPONENTS OR THE COORD
THEIR TERMINAL POINTS WHEN THEIR INITIAL POINTS, ARE AT THE ORIGIN OF THE COORDINA

Example 1 IF $\vec{a} = 2\mathbf{i} + 4\mathbf{j}$ AND $\vec{b} = 5\mathbf{i} + 3\mathbf{j}$, THEN FIND

I $\vec{a} + \vec{b}$ II $\vec{a} - \vec{b}$

Solution

I $\vec{a} + \vec{b} = (2\mathbf{i} + 4\mathbf{j}) + (5\mathbf{i} + 3\mathbf{j}) = (2+5)\mathbf{i} + (4+3)\mathbf{j} = 7\mathbf{i} + 7\mathbf{j}$

II $\vec{a} - \vec{b} = (2\mathbf{i} + 4\mathbf{j}) - (5\mathbf{i} + 3\mathbf{j}) = (2-5)\mathbf{i} + (4-3)\mathbf{j} = -3\mathbf{i} + \mathbf{j}$

NOTICE THAT THE TERMINAL POINTS OF THE VECTORS $\vec{a}$ AND $\vec{b}$ ARE AT (2, 4) AND (5, 3) RESPECTIVELY, WHILE THE TERMINAL POINT OF THE VECTOR $\vec{a} + \vec{b}$ IS AT (7, 7) WHICH CAN BE OBTAINED BY ADDING THE CORRESPONDING COORDINATES OF THE TERMINAL POINTS OF THE TWO VECTORS $\vec{a}$ AND $\vec{b}$. YOU MAY ALSO LOOK AT FIGURE 6.14 BELOW.

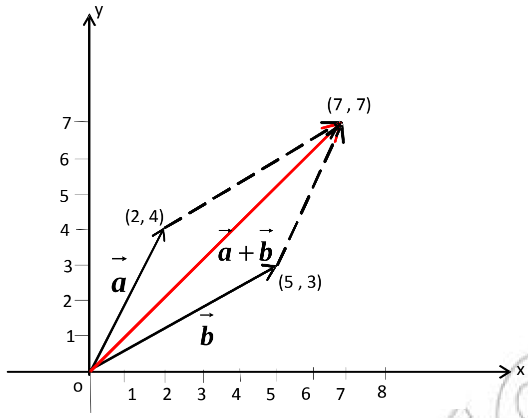


Figure 6.14

IN YOUR PREVIOUS STUDIES, YOU HAVE ALSO LEARNED ABOUT THE SCALAR OR DOT PROD
VECTORS. THAT IS, IF θ IS THE ANGLE BETWEEN THE TWO VECTORS $\vec{a}$ AND $\vec{b}$, THEN THE DOT OR SCALAR PRODUCT OF $\vec{a}$ AND $\vec{b}$ DENOTED BY $\vec{a}\cdot\vec{b}$ IS DEFINED AS:

$(\vec{a})\cdot(\vec{b}) = |\vec{a}||\vec{b}|\cos\theta$, WHERE $|\vec{a}|$ AND $|\vec{b}|$ ARE THE MAGNITUDES OF THE TWO VECTORS $\vec{a}$ AND $\vec{b}$ RESPECTIVELY.

Example 2 COMPUTE THE SCALAR PRODUCT OF THE VECTORS $\vec{a} = 3\mathbf{i} + 3\mathbf{j}$ AND $\vec{b} = 4\mathbf{i} + 0\mathbf{j}$.

Solution BY PICTURING A DIAGRAM, THE ANGLE BETWEEN THE TWO VECTORS IS 45°

THEN $|\vec{a}| = \sqrt{3^2+3^2} = 3\sqrt{2}$ AND $|\vec{b}| = \sqrt{4^2+0^2} = 4$

THUS $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta = 3\sqrt{2}(4)\cos 45^\circ = 12.$

OR

$\vec{a}\times\vec{b}$ = (3**i** +3**j**)·(4**i** +0**j**) = (3 × 4)**i**·**i** + (3 × 0)**i**·**j** + (3 × 4) **j**·**i** + (3 × 0) **j**·**j**

= 12 + 0 + 0 + 0 = 12.

The notion of vectors in space

JUST AS YOU WORKED WITH VECTORS ON A PLANE BY USING THE COORDINATES OF THEIR POINTS, YOU CAN HANDLE VECTORS IN A THREE DIMENSIONAL SPACE WITH THE HELP COORDINATES OF THE TERMINAL POINTS.

NOW, LET THE INITIAL POINT OF A VECTOR IN SPACE BE THE ORIGIN O OF THE COORDINAT AND LET ITS TERMINAL POINT BE AT A (x, y, z). THEN THE VECTOR $\overrightarrow{OA}$ CAN BE EXPRESSED AS THE SUM OF ITS THREE COMPONENTS IN THE DIRECTIONS OF THE x, y AND z AXIS, IN THE FORM:

$\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ WHERE $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ AND $\mathbf{k} = (0,0,1)$ ARE STANDARD UNIT VECTORS IN THE DIRECTIONS OF THE POSITIVE x-, THE POSITIVE y-AXIS AND POSITIVE z-AXIS, RESPECTIVELY. LOOK AT FIGURE 6.15 BELOW.

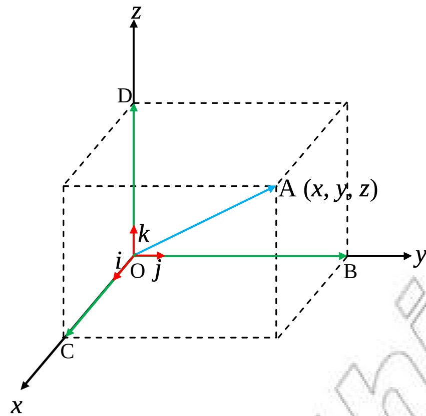


Figure 6.15

DO YOU OBSERVE THAT THE VECTOR $\overrightarrow{OA}$ IS THE SUM OF THE THREE PERPENDICULAR VECTORS $\overrightarrow{OC}, \overrightarrow{OB}$ AND $\overrightarrow{OD}$?

Example 3 IF THE INITIAL POINT OF A VECTOR IN SPACE IS AT THE ORIGIN AND ITS TERMINAL P HEAD IS AT P(3, 5, 4), SHOW THE VECTOR USING A COORDINATE SYSTEM AND IDEN ITS THREE PERPENDICULAR COMPONENTS IN THE DIRECTIONS OF THE THREE AXES.

Solution THE THREE COMPONENTS ARE THE VECTORS WITH COMMON INITIAL POINT O(0, 0 AND TERMINAL POINTS A(3, 0, 0) ON THE x-AXIS, B(0, 5, 0) ON THE y-AXIS AND C(0, 0, 4) ON THE z-AXIS AS SHOWN IN FIGURE 6.16

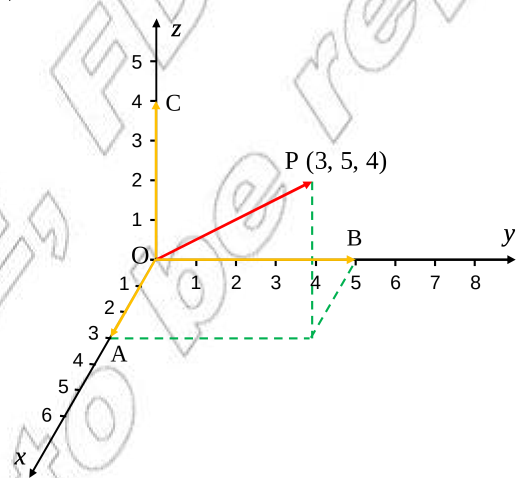


Figure 6.16

THAT MEANS, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ OR IN TERMS OF THE UNIT VECTORS,

$$(3, 5, 4) = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} = (3, 0, 0) + (0, 5, 0) + (0, 0, 4)$$

Addition and subtraction of vectors

JUST AS WITH VECTORS IN THE xy-PLANE, VECTORS IN SPACE CAN BE ADDED USING THE COORDINATES OF THEIR TERMINAL POINTS WHEN THEIR INITIAL POINTS ARE AT THE ORIGIN. IF $\vec{a}$ AND $\vec{b}$ ARE VECTORS IN SPACE WITH THEIR INITIAL POINTS AT THE ORIGIN AND THEIR TERMINAL POINTS AT (x_1, y_1, z_1) AND (x_2, y_2, z_2), RESPECTIVELY, THEN $\vec{a}+\vec{b}$ IS THE VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT (x_1+x_2, y_1+y_2, z_1+z_2).

HERE, IT IS ADVANTAGEOUS TO NOTE THAT A VECTOR $\vec{v}$ WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT POINT P(x, y, z) IN SPACE IS SHORTLY EXPRESSED AS $\vec{v}=\overrightarrow{OP}=(x,\ y,\ z)$.

THUS $\vec{v}=(3,\ 2,\ -4)$ IS THE VECTOR IN SPACE WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT (3, 2, –4).

Example 4 IF $\vec{a}=(1,3,2)$ AND $\vec{b}=(3,-1,4)$, FIND THE SUM VECTOR $\vec{a}+\vec{b}$.

Solution AS EXPLAINED ABOVE, THE SUM OF THE TWO VECTORS IS OBTAINED BY ADDING THE CORRESPONDING COORDINATES OF THE TERMINAL POINTS OF THE TWO VECTORS.

THAT IS, $\vec{a}+\vec{b}$ = (1, 3, 2) + (3, –1, 4) = (4, 2, 6) WHICH MEANS THAT $\vec{a}+\vec{b}$ IS THE VECTOR WHOSE INITIAL POINT IS THE ORIGIN AND WHOSE TERMINAL POINT IS AT (4,2,6).

SUBTRACTION OF A VECTOR FROM A VECTOR IS ALSO DONE IN A SIMILAR WAY. SO IF GIVEN TWO VECTORS $\vec{a}$ =(x_1, y_1, z_1) AND $\vec{b}$ =(x_2, y_2, z_2) THEN $\vec{a}-\vec{b}$ IS THE VECTOR $\vec{c}=(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$.

Example 5 IF $\vec{a}=(5,2,3)$ AND $\vec{b}=(3,1,4)$ THEN FIND $\vec{a}-\vec{b}$ AND $\vec{b}-\vec{a}$

Solution

I $\vec{a}-\vec{b}$ = (5, 2, 3) – (3, 1, 4) = (5–3, 2–1, 3–4) = (2, 1, –1)

THAT MEANS $\vec{a}-\vec{b}$ IS THE VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT (2, 1, –1) IN SPACE.

II $\vec{b}-\vec{a}$ = (3, 1, 4) – (5, 2, 3) = (3–5, 1–2, 4–3) = (–2, –1, 1).

DO YOU SEE THAT $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$?

Multiplication of a vector by a scalar

IF $\vec{a}$ = (x, y, z) THEN OBSERVE THAT

$2\vec{a}=\vec{a}+\vec{a}$ = (x, y, z) + (x, y, z) = ($x + x$, $y + y$, $z + z$) = ($2x$, $2y$, $2z$).

THUS, IT WILL BE REASONABLE TO ACCEPT THE FOLLOWING: IF $\vec{a}$ = (x, y, z) IS ANY VECTOR AND k IS ANY SCALAR (A NUMBER), THEN

$k\vec{a}$ = (kx, ky, kz) WHICH IS A VECTOR WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT (kx, ky, kz).

Example 6 IF $\vec{a} = (4, 2, 3)$, THEN

A $3\vec{a} = (12, 6, 9)$ **B** $-\vec{a} = (-4, -2, -3)$

C $-2\vec{a} = (-8, -4, -6)$ **D** $\frac{1}{2}\vec{a} = (2, 1, \frac{3}{2})$

Properties of addition of vectors

SINCE VECTOR ADDITION IS DONE USING THE COORDINATES OF THE TERMINAL POINTS OF THE ADDEND VECTORS, WHICH ARE REAL NUMBERS, YOU CAN EASILY VERIFY THE FOLLOWING PROPERTI ADDITION.

I Vector addition is commutative

FOR ANY TWO VECTORS $\vec{a} = (x_1, y_1, z_1)$ AND $\vec{b} = (x_2, y_2, z_2)$ IN SPACE,

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$. TO SEE THIS, LET US LOOK AT THE FOLLOWING.

$\vec{a} + \vec{b} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2) = (x_2+x_1, y_2+y_1, z_2+z_1)$ *Why?*

$= (x_2, y_2, z_2) + (x_1, y_1, z_1) = \vec{b} + \vec{a}$

II Vector addition is associative

FOR ANY THREE VECTORS $\vec{a}$, $\vec{b}$ AND $\vec{c}$ IN SPACE, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

III FOR TWO VECTORS $\vec{a}$ AND $\vec{b}$ AND ANY SCALAR k, YOU HAVE $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$.

Magnitude of a vector

AT THE BEGINNING OF THE DISCUSSION ABOUT VECTORS IN SPACE, IT WAS MENTIONED THAT A VECTOR IS USUALLY REPRESENTED BY AN ARROW, WHERE THE ARROW HEAD INDICATES THE DIREC LENGTH OF THE ARROW REPRESENTS THE MAGNITUDE OF THE VECTOR. THUS, TO FIND THE OF A VECTOR, IT WILL BE SUFFICIENT TO FIND THE DISTANCE BETWEEN THE INITIAL POIN TERMINAL POINT OF THE VECTOR IN THE COORDINATE SPACE.

FOR EXAMPLE, IF THE INITIAL POINT OF A VECTOR IS AT THE ORIGIN OF THE COORDINATE S THE TERMINAL POINT IS AT P(3, 2, 4) THEN THE MAGNITUDE OF THE VECTOR $\overrightarrow{OP}$ IS THE DISTANCE FROM O TO P. THIS IS, AS YOU KNOW, $\sqrt{3^2 + 2^2 + 4^2} = \sqrt{29}$

THUS, IN GENERAL, IF THE INITIAL POINT OF A VECTOR $\vec{v}$ IS AT THE ORIGIN AND ITS TERMINAL POINT IS AT A POINT Q(x, y, z) OR IF $\vec{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, THEN MAGNITUDE OF THE VECTOR $\vec{v}$, DENOTED BY $|\vec{v}|$ IS GIVEN BY $\sqrt{x^2 + y^2 + z^2}$. THAT IS,

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

IF THE INITIAL POINT OF $\vec{v}$ IS AT P(x_1, y_1, z_1) AND THE TERMINAL POINT AT Q(x_2, y_2, z_2), THEN

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Scalar or dot product of vectors in space

WHEN YOU WERE STUDYING VECTORS ON A PLANE, YOU SAW THAT THE DOT PRODUCT (SCALAR PRODUCT) OF TWO VECTORS $\vec{a}$ AND $\vec{b}$ WAS DEFINED BY:

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ WHERE θ IS THE ANGLE BETWEEN THE TWO VECTORS. IN PARTICULAR, FOR T

UNIT VECTORS $\vec{i}$ AND $\vec{j}$, YOU KNOW THAT $|\vec{i}|=|\vec{j}|=1$ AND FROM THE DEFINITION OF THE DOT PRODUCT, YOU EASILY SEE THAT $\vec{\mathbf{i}}.\vec{\mathbf{i}}=\vec{\mathbf{j}}.\vec{\mathbf{j}}=1$ AND $\vec{\mathbf{i}}.\vec{\mathbf{j}}=0=\mathbf{j}.\mathbf{i}$. SO, IF $\vec{a}=(x_1,y_1)$, AND $\vec{b}=(x_2,y_2)$

THE DOT PRODUCT $\vec{a}\cdot\vec{b}=x_1.x_2+y_1.y_2$ CAN BE VERIFIED VERY EASILY.

THE DOT (SCALAR) PRODUCT OF TWO VECTORS IN SPACE IS JUST AN EXTENSION OF THE DC OF VECTORS ON A PLANE. THAT MEANS IF $\vec{a}$ AND $\vec{b}$ ARE NOW TWO VECTORS IN SPACE, THE DOT (SCALAR) PRODUCT OF $\vec{a}$ AND $\vec{b}$ DENOTED BY $\vec{a}\cdot\vec{b}$ IS DEFINED AS:

$\vec{a}\cdot\vec{b}=|\vec{a}|.|\vec{b}|\cos\theta$ WHERE θ IS ONCE AGAIN THE ANGLE BETWEEN THE TWO VECTORS $\vec{a}$ AND $\vec{b}$.

OBSERVE THAT $\vec{a}\cdot\vec{b}$ IS A REAL NUMBER AND IN PARTICULAR IF

$\vec{a}=(x_1, y_1, z_1)=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ AND $\vec{b}=(x_2,y_2,z_2)=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$, YOU SEE THAT,

THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION ENABLES YOU TO FIND:

$\vec{a}\cdot\vec{b}=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}).(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})=x_1x_2+y_1y_2+z_1z_2$.

HERE IT IS IMPORTANT TO NOTE THAT FOR THE UNIT VECTORS $\vec{\mathbf{i}}, \vec{\mathbf{j}}$ AND $\vec{\mathbf{k}}$,

$\vec{\mathbf{i}}.\vec{\mathbf{i}}=\vec{\mathbf{j}}.\vec{\mathbf{j}}=\vec{\mathbf{k}}.\vec{\mathbf{k}}=1$ WHILE $\vec{\mathbf{i}}.\vec{\mathbf{j}}=\vec{\mathbf{i}}.\vec{\mathbf{k}}=\vec{\mathbf{k}}.\vec{\mathbf{j}}=0$ THE REASON BEING THAT THE MAGNITUDE OF A UNIT VECTOR IS ONE, $\cos 90^{o}=0$ AND $\cos 0^{o}=1$.

Example 7 IF $\vec{a}=(2, 3, -1)$ AND $\vec{b}=(-1, 0, 2)$, THEN FIND THE SCALAR (DOT) PRODUCT OF $\vec{a}$ AND $\vec{b}$.

Solution $\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2=2(-1)+3(0)+(-1)(2)=-4$

Example 8 IF $\vec{a}=(2, 0, 2)$ AND $\vec{b}=(0, 3, 0)$ FIND THEIR DOT PRODUCT.

Solution $\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2=2(0)+0(3)+2(0)=0$

OBSERVE THAT (2, 0, 2) AND (0, 3, 0) ARE PERPENDICULAR VECTORS I.E. THE ANGLE BETV THEM IS 90^{o}.

Exercise 6.5

1 CALCULATE THE MAGNITUDE OF EACH OF THE FOLLOWING VECTORS.

A (–1, 3, 0) B (3, 1, –1) C $\left(\frac{1}{2},\frac{3}{2},\frac{4}{5}\right)$

2 FIND THE SCALAR (DOT) PRODUCT OF EACH OF THE FOLLOWING PAIR OF VECTORS.

A (2, –3, 1) AND (1, 0, 4) B (–5,0,1) AND (1,–3,–2)

C (–2, 2, 0) AND (0, 0,–1) D (0,0,3) AND (0,0,3)

Angle between two vectors in space

FOR TWO VECTORS $\vec{a}$ AND $\vec{b}$ WITH INITIAL POINT AT THE ORIGIN, THEIR DOT PRODUCT IS DEFINED BY

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ WHERE θ IS THE ANGLE BETWEEN THE TWO VECTORS, ASSUMING THAT BOTH VECTORS HAVE THE SAME INITIAL POINT AT THE ORIGIN. THEN SOLVING FOR $\cos\theta$, YOU CAN REWRITE THE ABOVE EQUATION IN THE FORM:

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

HENCE THE ANGLE θ BETWEEN THE TWO VECTORS CAN BE OBTAINED USING THIS LAST FORMULA PROVIDED THE VECTORS ARE NON-ZERO.

Example 9 FIND THE ANGLE BETWEEN THE VECTORS $\vec{a} = (2, 0, 0)$ AND $\vec{b} = (0, 0, 3)$.

Solution $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

BUT $\vec{a}\cdot\vec{b} = 2(0) + (0)(0) + 0(3) = 0$

$|\vec{a}| = \sqrt{2^2+0^2+0^2} = \sqrt{2}$ AND $|\vec{b}| = \sqrt{0^2+0^2+3^2} = 3$

THEREFORE $\cos\theta = \dfrac{0}{2(3)} = 0 \Rightarrow \theta = 90^{\circ}$

NOTICE THAT, THE VECTOR (2, 0, 0) IS ALONG THE x-AXIS WHILE THE VECTOR (0, 0, 3) IS ALONG THE z-AXIS AND THE TWO AXES ARE PERPENDICULAR TO EACH OTHER.

Example 10 FIND THE ANGLE BETWEEN THE VECTORS $\vec{a} = (1,0,1)$ AND $\vec{b} = (1,1,0)$.

Solution $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

BUT $\vec{a}\cdot\vec{b} = 1(1) + 0(1) + 1(0) = 1$

$|\vec{a}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$ AND $|\vec{b}| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

THEREFORE $\cos\theta = \dfrac{1}{\sqrt{2}.\sqrt{2}} = \dfrac{1}{2} \Rightarrow \theta = 60^{O}$

NOTICE THAT THE VECTOR $\vec{a}$ IS ON THE xz-PLANE AND $\vec{b}$ IS ON THE xy-PLANE, EACH FORMING A 45^{O} ANGLE WITH THE x-AXIS.

Exercise 6.6

1 IF THE VECTORS $\vec{a}, \vec{b}, \vec{v}$ AND $\vec{u}$ ARE AS GIVEN BELOW:

$\vec{a} = (1, 3, 2), \vec{b} = (0, -3, 4), \vec{v} = (-4, 3, -2), \vec{u} = (\frac{1}{2}, 0, -3)$, THEN FIND EACH OF THE FOLLOWING VECTORS.

- **A** $\vec{a} + \vec{b}$
- **B** $\vec{b} + \vec{a}$
- **C** $\vec{a} - \vec{b}$
- **D** $\vec{b} - \vec{a}$
- **E** $\vec{a} + \vec{b} + \vec{v}$
- **F** $\vec{b} + \vec{v} - \vec{u}$
- **G** $\vec{a} + \vec{b} + \vec{v} + \vec{u}$

2 IF THE VECTORS $\vec{a}, \vec{b}, \vec{v}$ AND $\vec{u}$ ARE AS GIVEN IN QUESTION 1 ABOVE, THEN FIND

- **A** $3\vec{a}$
- **B** $-4\vec{b}$
- **C** $2\vec{a} + 3\vec{b}$
- **D** $3\vec{b} - \frac{1}{2}\vec{a} + 2\vec{v}$

3 VERIFY THAT VECTOR ADDITION IS ASSOCIATIVE. THAT IS, FOR ANY THREE VECTORS $\vec{a}$, $\vec{b}$ AND $\vec{c}$ IN SPACE, SHOW THAT $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

4 FOR ANY TWO VECTORS $\vec{a}$ AND $\vec{b}$ IN SPACE AND ANY SCALAR k, SHOW THAT

$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

5 WRITE EACH OF THE FOLLOWING VECTORS AS A SUM OF ITS COMPONENTS USING THE STANDARD UNIT VECTORS i, j AND k.

- **A** $(-4, 3. -2)$
- **B** $(1, -3, \sqrt{2})$
- **C** $(3, 5, -7)$
- **D** $(0, 0, 3)$

6 SHOW EACH OF THE FOLLOWING VECTORS IN THE COORDINATE SPACE USING AN ARROW THAT STARTS FROM THE ORIGIN.

- **A** $\vec{a} = (3, 3, 3)$
- **B** $\vec{b} = (-3, 3, 4)$
- **C** $\vec{c} = (2, -3, -3)$

7 CALCULATE THE MAGNITUDE OF EACH VECTOR IN QUESTION 6 ABOVE.

8 FIND THE SCALAR (DOT) PRODUCT OF EACH OF THE FOLLOWING PAIRS OF VECTORS.

- **A** (1, 0, 1) AND (2, 2, 0)
- **B** (–2, 5, 1) AND (1 , –1, –2)
- **C** $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ AND $(-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2})$
- **D** (1, 0, 1) AND (–1, 1, 0)
- **E** (5, 0, 0) AND (0, –5, 0)
- **F** (2, 2, 2) AND (–1, –1, –1)

9 FIND THE ANGLE BETWEEN EACH OF THE FOLLOWING PAIRS OF VECTORS.

- **A** (2,0,1) AND (0,–1,0)
- **B** (1,1,1) AND (1,0,1)
- **C** (–1,1,1) AND (2,2,2)

Key Terms

angle between two vectors
concurrent edges of a rectangular box
coordinate planes
coordinate system
diagonal of a rectangular box
dot product of vectors
magnitude of a vector
mutually perpendicular lines
octants
ordered triples of real numbers
reference axes
unit vectors
vector in space

Summary

1 THREE MUTUALLY PERPENDICULAR LINES IN SPACE DIVIDE THE SPACE INTO eight octants.

2 IF (x, y, z) ARE THE COORDINATES OF A POINT P IN SPACE, THEN
- ✓ x IS THE DIRECTED DISTANCE OF THE POINT P FROM THE yz-PLANE,
- ✓ y IS THE DIRECTED DISTANCE OF THE POINT P FROM THE xz-PLANE,
- ✓ z IS THE DIRECTED DISTANCE OF THE POINT P FROM THE xy-PLANE.

3 THERE IS A ONE TO ONE CORRESPONDENCE BETWEEN THE SET OF ALL POINTS OF THE SP
THE SET OF ALL ORDERED TRIPLES OF REAL NUMBERS.

4 THE DISTANCE BETWEEN TWO POINTS $P(x, y, z)$ AND $Q(a, b, c)$ IN SPACE IS GIVEN BY

$d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. THUS THE DISTANCE OF A POINT

$P(x, y, z)$ FROM THE ORIGIN IS $\sqrt{x^2 + y^2 + z^2}$.

5 THE midpoint OF A LINE SEGMENT WITH END POINTS $A(x, y, z)$ AND $B(a, b, c)$ IN SPACE IS THE POINT $M\left(\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2}\right)$

6 THE equation of a sphere WITH CENTRE AT $C(x_1, y_1, z_1)$ AND RADIUS r IS GIVEN BY

$$(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = r^2$$

IN PARTICULAR, IF THE CENTRE IS AT THE ORIGIN AND r IS THE RADIUS, THE EQUATION BECOMES $x^2 + y^2 + z^2 = r^2$, WHERE (x, y, z) ARE COORDINATES OF ANY POINT P ON THE SURFACE OF THE SPHERE.

7 IN SPACE, IF THE INITIAL POINT OF A VECTOR IS AT THE ORIGIN O OF THE COORDINATE
AND ITS TERMINAL POINT IS AT A POINT $A(x, y, z)$, THEN IT CAN BE EXPRESSED AS THE SUM OF ITS THREE COMPONENTS IN THE DIRECTIONS OF THE THREE AXES AS:

$\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ WHERE $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ AND $\mathbf{k} = (0, 0, 1)$ ARE THE standard unit vectors IN THE DIRECTIONS OF THE POSITIVE x, POSITIVE y AND POSITIVE z-AXES RESPECTIVELY.

8 IN SPACE IF VECTORS $\vec{a}$ AND $\vec{b}$ HAVE THEIR INITIAL POINT AT THE ORIGIN AND THEIR TERMINA POINTS AT A(x_1, y_1, z_1) AND B(x_2, y_2, z_2) RESPECTIVELY, THEN THE sum $\vec{a}+\vec{b}$ IS A VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT C($x_1 + x_2$, $y_1 + y_2$, $z_1 + z_2$).

SIMILARLY, THE difference $\vec{a}-\vec{b}$ IS THE VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT D($x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$).

9 IF THE INITIAL POINT OF A VECTOR $\vec{a}$ IS AT THE ORIGIN AND THE TERMINAL POINT IS AT P(x, y, z), THEN FOR ANY CONSTANT k, THE PRODUCT $k\vec{a}$ IS A VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN AND TERMINAL POINT IS AT Q(kx, ky, kz).

10 VECTOR ADDITION IS COMMUTATIVE AND ALSO ASSOCIATIVE.

11 MULTIPLICATION OF A VECTOR BY A SCALAR IS DISTRIBUTIVE OVER VECTOR ADDITIO FOR VECTORS $\vec{a}$ AND $\vec{b}$ AND A SCALAR k, $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$.

12 THE MAGNITUDE OF A VECTOR $\vec{a}$ WITH INITIAL POINT AT THE ORIGIN AND TERMINAL POINT AT P(x, y, z) IS GIVEN BY:

$$|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}$$

13 THE dot (scalar) product OF TWO VECTORS $\vec{a}$ AND $\vec{b}$ IS GIVEN BY $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|.\cos\theta$, WHERE θ IS THE ANGLE BETWEEN THE TWO VECTORS $\vec{a}$ AND $\vec{b}$.

14 IF θ IS THE ANGLE BETWEEN TWO VECTORS $\vec{a}$ AND $\vec{b}$, THEN $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$.

Review Exercises on Unit 6

1 LOCATE THE POSITION OF EACH OF THE FOLLOWING POINTS AND FIND THE DISTANCE POINT FROM THE ORIGIN.

A O(0,0, –3) **B** P(0, –1, 2) **C** Q(3,–1,4)

D R(–1,–2,–3) **E** S(–3, –2, 3) **F** T(–3, –3, –4)

G U(4, 3, –2) **H** V $\left(0, -\frac{3}{2}, \frac{5}{2}\right)$

2 FIND THE DISTANCE OF THE POINT P (4, –3, 5) FROM THE

A ORIGIN **B** xy-PLANE **C** xz-PLANE **D** yz-PLANE

E x-AXIS **F** y-AXIS **G** z-AXIS

3 FOR EACH OF THE FOLLOWING PAIRS OF POINTS, FIND THE DISTANCE FROM A TO B AN FIND THE MIDPOINT OF $\overline{AB}$.

A A (–1,2,3) AND B (0,–1,1) **B** A (3,–1,1) AND B (–1, 0,1)

C A (2,0,–3) AND B (2,–1,3) **D** A (0,0,–4) AND B (4,0,0)

E $A(-2, -1, -3)$ AND $B(2, 2, 1)$

F $A\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ AND $B(10, 0, 11)$

G $A(\sqrt{2}, 5, 0)$ AND $B\left(0, \frac{1}{2}, \sqrt{3}\right)$

H $A(0, 0, -2)$ AND $B(0, 0, 5)$

4 FIND THE MIDPOINT OF THE LINE SEGMENT WHOSE END POINTS ARE:

A A (0, 0, 0) AND B (4, 4, 4)

B C (–2, –2, –2) AND D (2, 2, 2)

C P (6, 0, 0) AND Q (0, 4, 0)

D $R(2\sqrt{2}, -4, 0)$ AND $S(-2\sqrt{2}, 0, -5)$

5 SHOW THAT A (0,4,4), *B* (2,6,5) AND *C* (1,4,3) ARE VERTICES OF AN ISOSCELES TRIANGLE.

6 DETERMINE THE NATURE OF $\triangle ABC$ USING DISTANCES, IF THE VERTICES ARE AT:

A $A(2,-1,7)$, $B(3,1,4)$ AND $C(5, 4,5)$

B $A(0,0,3)$, $B(2,8,1)$ AND $C(-9,6,18)$

C $A(1,0,-3)$, $B(2,2,0)$ AND $C(4,6,6)$

D $A(5, 6, -2)$, $B(6, 12, 9)$ AND $C(2, 4, 2)$

7 MAKE A THREE DIMENSIONAL SKETCH SHOWING EACH OF THE FOLLOWING VECTORS WITH INITIAL POINT AT THE ORIGIN.

A $\vec{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$,

B $\vec{b} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$,

C $\vec{c} = -3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$,

D $\vec{d} = 4\mathbf{j} - 7\mathbf{k}$.

8 USING THE VECTORS IN QUESTION 7 ABOVE, CALCULATE EACH OF THE FOLLOWING.

A $\vec{a} + \vec{b}$ **B** $2\vec{a} - \vec{c}$ **C** $\vec{b} + \vec{c} + \vec{d}$ **D** $2\vec{a} - 3\vec{b} + \vec{c}$

9 CALCULATE THE MAGNITUDE OF EACH OF THE VECTORS IN QUESTION 7 ABOVE.

10 A SPHERE HAS CENTRE AT C(–1, 2, 4) AND DIAMETER AB, WHERE A IS AT (–2, 1, 3). FIND THE COORDINATES OF B, THE RADIUS AND THE EQUATION OF THE SPHERE.

11 DECIDE WHETHER OR NOT EACH OF THE FOLLOWING IS AN EQUATION OF A SPHERE. IF IT IS AN EQUATION OF A SPHERE, DETERMINE ITS CENTRE AND RADIUS.

A $x^2 + y^2 + z^2 - 2y = 4$

B $x^2 + y^2 + z^2 - x + 2y - 3z + 4 = 0$

C $x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$

12 CALCULATE THE SCALAR (DOT) PRODUCT OF EACH OF THE FOLLOWING PAIRS OF VECTORS.

A $\vec{a} = (3,2,-4)$ AND $\vec{b} = (3,-2,7)$

B $\vec{c} = (-1, 6, 5)$ AND $\vec{d} = (10,3,1)$

C $\vec{p} = (2, 5, 6)$ AND $\vec{q} = (6, 6, -7)$

D $\vec{a} = (7, 8, 9)$ AND $\vec{b} = (5, -9, 5)$

13 FOR EACH PAIR OF VECTORS GIVEN IN QUESTION 12 ABOVE, FIND THE COS θ, WHERE θ IS THE ANGLE BETWEEN THE VECTORS.